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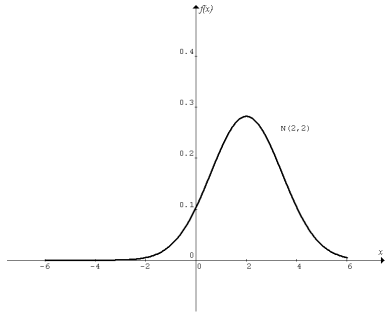
The normal distribution, an example of continuous probability distributions

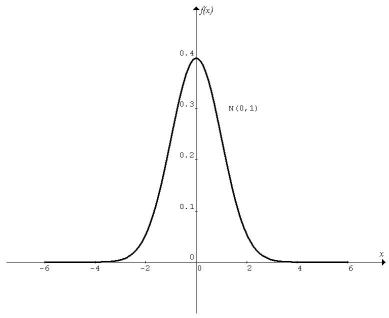
The normal distribution $N(\mu,\sigma^2)$ *has the following probability density function:* $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ *, where the mean* $\mu\in R$ *and the standard deviation* $\sigma>0$ *are the parameters of the normal distribution.*

Note: $N(\mu,\sigma^2)$ *is the notation used in the study design,* σ^2 *is the variance. You may find in some textbooks the notation* $N(\mu,\sigma)$ *is used.*

A normal distribution is bell-shaped and it is symmetrical about its mean μ*. At* $x=\mu$ *,* $f(\mu)=\frac{1}{\sigma\sqrt{2\pi}}$ *is the maximum,* $\therefore \mu$ *is also the mode. Since* $\Pr(X<\mu)=0.5$ *and* $\Pr(X>\mu)=0.5$ *due to the symmetry about the mean,* $\therefore \mu$ *is also the median of a normal distribution.*

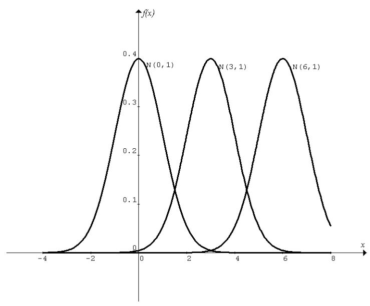


If $\mu=0$ *and* $\sigma=1$*, then* $N(0,1)$ *is called the standard normal distribution. Its random variable is usually denoted as Z.*

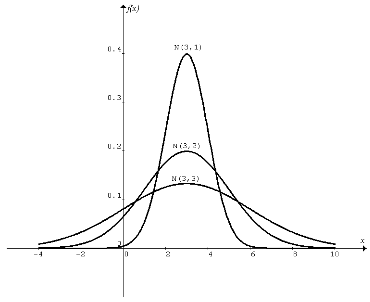


Effects of changing μ and σ on the graph of $N(\mu,\sigma^2)$

Increasing (decreasing) the value of μ *causes the graph to translate to the right (left).*



Increasing (decreasing) the value of σ *causes the graph to spread out (contract) and lowers (raises) the maximum value of* $f(x)$.



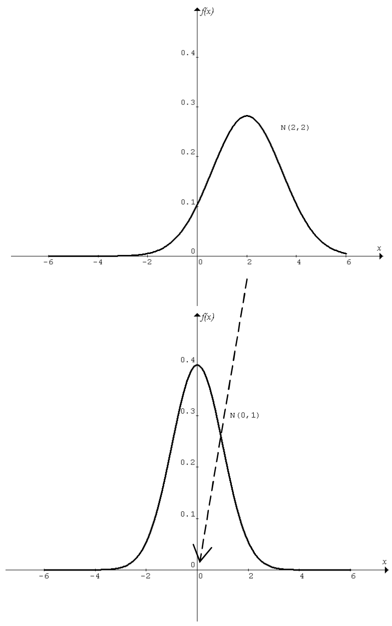
Transforming a normal distribution to the standard normal distribution

A value of any random variable X with normal distribution $N(\mu,\sigma^2)$ *can be matched up with a value of the random variable Z with the standard normal distribution* $N(0,1)$ *by the relationship* $z=\frac{x-\mu}{\sigma}$*, such that* $\Pr(X<x)=\Pr\left(Z<\frac{x-\mu}{\sigma}\right)$.

The values of Z are called the standardised values of X, or simply z-values or z-scores.

Example 1 Compare $N(2,2)$ with the standard normal distribution.

$\mu=2$, $\sigma^2=2, \therefore \sigma=\sqrt{2}$.



For $x=2$, $z=\frac{2-2}{\sqrt{2}}=0$. $\therefore \Pr(X<2)=\Pr(Z<0)=0.5$.

For $x=0$, $z=\frac{0-2}{\sqrt{2}}=-1.414$.

$\therefore \Pr(X<0)=\Pr(Z<-1.414)=0.079$

Example 2

(a) Write down the probability density function $f(x)$ for $N(2.5,2.25)$.

(b) Determine $\Pr(X<3)$ by (i) finding the area under the graph of $f(x)$, (ii) by CAS, (iii) changing to the standard normal and then by CAS.

(c) Determine $\Pr(1<X<3)$ by working through (i), (ii), (iii).

(d) Show that $\Pr(\mu-2\sigma<X<\mu+2\sigma)\approx 0.95$.

(a) $\mu = 2.5$, $\sigma = \sqrt{2.25} = 1.5$, $f(x) = \frac{1}{1.5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2.5}{1.5}\right)^2}$

(b) i. $\Pr(X < 3) = \int_{-\infty}^{3} f(x)dx = 0.63$.

ii. $\Pr(X < 3) = 0.63$ by Normal Cdf.

iii. $z = \frac{3-2.5}{1.5} = \frac{1}{3}$, $\Pr(X < 3) = \Pr\left(Z < \frac{1}{3}\right) = 0.63$.

(c) i. $\Pr(1 < X < 3) = \int_{1}^{3} f(x)dx = 0.47$.

ii. $\Pr(1 < X < 3) = 0.47$ by Normal Cdf.

iii. $z = \frac{1-2.5}{1.5} = -1$, $\mu = 0$, $\sigma = 1$

$\Pr(1 < X < 3) = \Pr\left(-1 < Z < \frac{1}{3}\right) = 0.47$ by Normal Cdf.

(d) $\mu - 2\sigma = 2.5 - 2\times1.5 = -0.5$, $\mu + 2\sigma = 2.5 + 2\times1.5 = 5.5$,
$\Pr(-0.5 < X < 5.5) \approx 0.95$ by Normal Cdf.

Example 3 Given $X \sim N(20,36)$, find $\Pr(8 \le X \le 26)$ without using CAS.

$\Pr(8 \le X \le 26) = \Pr(-2 \le Z \le 1) = \Pr(-2 \le Z \le 0) + \Pr(0 \le Z \le 1)$
$\approx \frac{0.95}{2} + \frac{0.68}{2} = 0.815$

Finding the value *x* of random variable *X* from the given probability of $X \le x$

By CAS Inverse Normal
If $\Pr(X > x)$ *is given, let* $\Pr(X > x) = 1 - \Pr(X \le x)$ *and then use CAS Inverse Normal to find* $\Pr(X \le x)$

Example 3 *X* has a normal distribution with $\mu = 80$, $\sigma^2 = 25$. Find (a) *x* if $\Pr(X < x) = 0.8$ (b) *x* if $\Pr(X > x) = 0.8$
(c) $c > 0$ if $\Pr(\mu - c < X < \mu + c) = 0.8$.

(a) $\sigma = 5$, $\Pr(X < x) = 0.8$, $x = 84.2$ by CAS Inverse Normal.
(b) $\Pr(X > x) = 0.8$, $1 - \Pr(X \le x) = 0.8$, $\Pr(X \le x) = 0.2$,
$x = 75.79$ by CAS Inverse Normal.
(c) $\Pr(\mu - c < X < \mu + c) = 0.8$, $\therefore \Pr(\mu - c < X < \mu) = 0.4$,
$\Pr(X < \mu - c) = 0.1$, $\mu - c = 73.59$ by CAS Inverse Normal,
$c = 80 - 73.59 = 6.41$

Finding μ and σ of random variable *X* from the probability of $X \le x$

Use $z = \frac{x - \mu_X}{\sigma_X}$ *to convert x to z in order to incorporate* μ_X *and* σ_X *in the equation(s). By CAS Inverse Normal with standard* $\mu_Z = 0$ *and* $\sigma_Z = 1$*, equation(s) involving* μ_X *and/or* σ_X *can be found. Solve the equation(s) to find* μ_X *and/or* σ_X.

Example 4 The height of a population is normally distributed with $\mu = 175$ cm. If 2% of the population is over 190 cm, find the standard deviation of the random variable *X* which is the height of the population in cm.

Given $\Pr(X > 190) = 0.02$, $\Pr(X < 190) = 1 - 0.02 = 0.98$,
$\Pr\left(Z < \frac{190-175}{\sigma}\right) = 0.98$.

Hence $\frac{15}{\sigma} = 2.0537$ by CAS Inverse Normal, $\therefore \sigma = 7.3$.

Example 5 The body-weight of an individual in a population is normally distributed. If 5% of the population is over 80 kg and 3% is less than 20 kg, find the mean and standard deviation of the weight of the population.

Given $\Pr(X > 80) = 0.05$ and $\Pr(X < 20) = 0.03$,
$\therefore \Pr(X < 80) = 1 - 0.05 = 0.95$,
$\therefore \Pr\left(Z < \frac{80-\mu}{\sigma}\right) = 0.95$ and $\Pr\left(Z < \frac{20-\mu}{\sigma}\right) = 0.03$.

$\frac{80-\mu}{\sigma} = 1.645$ and $\frac{20-\mu}{\sigma} = -1.881$ by CAS Inverse Normal

Solve simultaneously,
$80 - \mu = 1.645\sigma$ ....(1) $20 - \mu = -1.881\sigma$ ....(2)
$\sigma \approx 17$ and $\mu \approx 52$

Example 6 The distance in metres that a javelin is thrown by a competitor follows a normal distribution with a mean of 80.80 and a standard deviation of 4.50. (a) Find the probability that a javelin is thrown more than 85.00 m. (b) In five throws what is the probability that the competitor throws more than 85.00 m in more than 2 occasions? (c) Find the expected number of throws by the competitor exceeding 85.00 m in 5 throws.

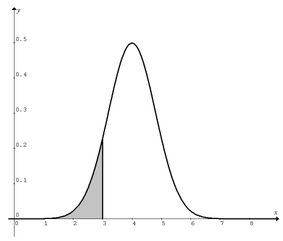
(a) $N(80.80, 4.50^2)$, $\Pr(X > 85.00) = 0.175$ by Normal Cdf.
(b) $Bi(5, 0.175)$, $\Pr(X > 2) = 0.04051$ by Binomial Cdf.
(c) $Bi(5, 0.175)$, $\mu = E(X) = np = 5\times0.175 = 0.88$.

Example 7 The weight in grams of a potato in a crate is normally distributed with $\mu = 220$ and $\sigma = 10$. Find (a) the probability that a potato weighs more than 215 grams given that it is not less than 210 grams, (b) the probability that a potato weighs less than 215 grams given that it is not less than 210 grams.

(a) $\Pr(X > 215 \mid X \ge 210) = \frac{\Pr(X > 215 \cap X \ge 210)}{\Pr(X \ge 210)}$
$= \frac{\Pr(X > 215)}{\Pr(X \ge 210)} = 0.82$ by Normal Cdf.

(b) $\Pr(X < 215 \mid X \ge 210) = 1 - \Pr(X > 215 \mid X \ge 210)$
$= 1 - 0.82 = 0.18$.

Example 8 The lifetime X of a brand of car battery is normally distributed with a mean of 4 years and a standard deviation of 0.8 of a year. The manufacturer wishes to guarantee the battery for 3 years. What percentage of the batteries will the manufacturer have to replace under the guarantee?



$\Pr(X < 3) = 0.106$ by Normal Cdf. .: 10.6%

Example 9 Bolts with a diameter of 0.280 ± 0.002 cm are required for a certain job. A manufacturer produces bolts with diameters normally distributed with $\mu = 0.281$ and $\sigma = 0.001$ cm. Find the proportion of bolts meeting the specification.

Let random variable *X* be the diameter of a bolt.
$\Pr(0.278 \le X \le 0.282) = 0.84$ by Normal Cdf.

Questions: Next page

1. The normal distribution of a random variable X has a probability density function given by

$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$. Find μ and σ of X with

$f(x)=0.7979e^{-2(x-5)^2}$.

2. Without using CAS, find the following probabilities, correct to 2 decimal places.

(a) $\Pr(Z>1)$

(b) $\Pr(Z<2)$

(c) $\Pr(0\le Z<2)$

(d) $\Pr(-1\le Z\le 2)$

(e) $\Pr(-2\le Z\le 1)$

3. X is normally distributed with $\mu=72$ and $\sigma=8$. Without using CAS find (a) $\Pr(X>80)$ (b) $\Pr(64<X<72)$ (c) $\Pr(X<64 \mid X<72)$ (d) the median of X.

4. The times (in minutes) taken by students to complete a test are normally distributed with mean 45 minutes and standard deviation 5 minutes. Find the proportion of students who complete the test in less than 42 minutes.

5. The heights of the children in a queue at a Lunar Park ride are normally distributed with mean 130 cm and standard deviation 2.7 cm. Only 90% of the children are allowed to go on the ride because they are tall enough. Find the minimum acceptable height.

6. X has a normal distribution with mean 5 and variance 9 and Z has the standard normal distribution. Find b such that $\Pr(X>7)=\Pr(Z<b)$.

7. Let X be normal with mean 3.6 and variance 0.01. Find c such that (a) $\Pr(X\le c)=50\%$ (b) $\Pr(X>c)=10\%$ (c) $\Pr(-c<X-3.6\le c)=99.9\%$

8. The thickness X (in mm) of timber boards in production is normal with mean 10.00 mm and standard deviation 0.02 mm. Determine the % of defective boards to be expected, assuming defective boards are (a) boards thinner than 9.97 mm (b) thicker than 10.05 mm (c) boards that deviate more than 0.03 mm from 10.00 mm.

9. Refer to Q8c. (a) What % of defective boards can we expect if $\sigma=0.01$ mm? (b) What value should σ have in order that the % of defective boards reduces to 6%?

10. Refer to Q8. Find c such that the expected % of defective boards (those outside the interval $[10-c,10+c]$) will not be greater than 5%.

11. Refer to Q8 and Q9. After making some changes in the production process. The mean is shifted from 10 mm to 10.01 mm. Find the expected % of defective boards (those outside the interval $[10-c,10+c]$).

Numerical, algebraic and worded answers.

1. 5, 0.5 2. (a) 0.16 (b) 0.98 (c) 0.48 (d) 0.82 (e) 0.82
3. (a) 0.16 (b) 0.34 (c) 0.32 (d) 72 4. 0.2743
5. 126.5 cm 6. -0.67
7. (a) 3.6 (b) 3.7282 (c) 0.3291
8. (a) 6.68% (b) 0.62% (c) 13.36%
9. (a) 0.27% (b) 0.0160 mm
10. $c\ge 0.0392$
11. 8%