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Free maths self-help revision

Expected value (mean)

The common measure of the centre of a distribution of values of a random variable X is the ***expected value*** *(the* ***mean****, or* ***expectation****) of X. It is denoted as* $E(X)$, $\overline{X}$ *or* μ_X *. If a random experiment is repeated many times and the average value of the random variable is calculated, it approaches the expected value as the number of repetitions increases.*

Calculation of expected value of a discrete random variable

$\mu_X = E(X) = \sum x \Pr(X = x)$

Example 1 Find the expectation of the following distribution.

x	-2	0	1	10
$\Pr(X = x)$	0.26	0.33	0.38	0.03

$E(X) = -2(0.26) + 0(0.33) + 1(0.38) + 10(0.03) = 0.16$

Example 2 Find the mean of X with probability distribution given by $\Pr(X = x) = {}^3C_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x}$ for $x = 0, 1, 2, 3$.

$$E(x) = 0\left({}^3C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^3\right) + 1\left({}^3C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^2\right) + 2\left({}^3C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1\right)$$

$$+3\left({}^3C_0\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0\right) = 0.75.$$

Note: It is easier to set up the probability distribution table and use it to calculate $E(X)$. See example 9 under **Variance and standard deviation**.

Example 3 A fair die is rolled. A player who bets a dollar on a number receives \$6 if that number is the uppermost number, otherwise the dollar is lost. What does the player expect to win on a roll of the die? Interpret the answer.

Let X be the winning amount.

x	-1	$+5$
$\Pr(X = x)$	5/6	1/6

$E(X) = -1\times\frac{5}{6} + 5\times\frac{1}{6} = 0$. In the long run the player wins/loses nothing. .: it is a fair game.

Example 4 An insurance company sells to 2000 individuals policies that protect their cars against theft for a period of a year. Based on an average cost of \$10 000 to replace a stolen car, the company determines that it will break even if the probability of a theft during the policy period is 0.01. Assume that the probability of more than one theft per individual is 0. (a) What should be the selling price of the insurance policy? (b) If the probability of theft is actually 0.005, what is the company's expected gain per policy for the same selling price in (a)?

(a) Let \$$p$ be the selling price of each policy, and X be the gain amount per policy in dollars.

x	$p - 10\,000$	$+p$
$\Pr(X = x)$	0.01	0.99

Break even, $E(X) = 0$, $(p - 10000)\times 0.01 + p\times 0.99 = 0$,

$p = 100$ dollars

(b)

x	$100 - 10\,000$	$+100$
$\Pr(X = x)$	0.005	0.995

$E(X) = (100 - 10000)\times 0.005 + 100\times 0.995 = 50$ dollars

Median and mode

Other measures of the centre of a distribution are the median and mode of X.

The median m *of X has the following properties:*

$\Pr(X \le m) \ge 0.5$ *and* $\Pr(X \ge m) \ge 0.5$, *i.e. the probability at or less than m is at least 0.5 and the probability at or greater than m is at least 0.5.*

The mode of X is the value of X that has the highest probability.

Determination of median and mode

Example 5

x	-1	0	1.5	1.8	3
$\Pr(X = x)$	0.03	0.12	0.3	0.2	0.35

Median $m = 1.8$ because it satisfies the properties $\Pr(X \le m) \ge 0.5$ and $\Pr(X \ge m) \ge 0.5$. The mode of X is 3.

Example 6

x	1	1.2	1.5	1.8	2.3
$\Pr(X = x)$	0.03	0.42	0.05	0.2	0.3

$\Pr(X \le 1.5) = 0.5$ and $\Pr(X \ge 1.8) = 0.5$. The median lies between 1.5 and 1.8. It is taken as midway between 1.5 and 1.8, i.e. $m = \frac{1.5 + 1.8}{2} = 1.65$. The mode of X is 1.2.

Example 7

x	3	5.2	7.5	8	11
$\Pr(X = x)$	0.11	0.32	0.05	0.32	0.2

The median is 8, and the modes are 5.2 and 8.

Expected value of the sum of two random variables

$E(X + Y) = E(X) + E(Y)$

Variance and standard deviation

The spread of the probability distribution is measured by a number called the ***variance*** *of X, and it is denoted as* $Var(X)$.

Another number called the ***standard deviation*** *of X is often used instead of the variance to measure the spread. It is denoted by* $sd(X)$, *or* σ_X.

$\sigma_X = \sqrt{Var(X)}$

In many cases, if the random experiment is repeated many times, ~68% of the recorded values of X lie within one σ_X *from* μ_X, *and ~95% of the recorded values of X lies within* $2\sigma_X$ *from* μ_X.

$\therefore$ *in a single trial of the random experiment,*

$\Pr(\mu - \sigma \le X \le \mu + \sigma) \approx 0.68$, *and*

$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$.

Calculation of $Var(X)$ **and** σ_X

The variance of discrete random variable X is defined as

$Var(X) = E\left((X - \mu_X)^2\right) = \sum (x - \mu_X)^2 \Pr(X = x)$.

It can be shown that the above definition is equivalent to

$Var(X) = E(X^2) - (\mu_X)^2 = \sum x^2 \Pr(X = x) - (\mu_X)^2$.

The second formula is easier to use than the first in calculating variance.

Example 8 Calculate the variance and standard deviation of the following discrete probability distribution.

x	-2	0	1	10
$\Pr(X = x)$	0.26	0.33	0.38	0.03

Calculate the mean of X, $\mu_X = E(X) = 0.16$ (See example 1).

$Var(X) = (-2)^2(0.26) + 0^2(0.33) + 1^2(0.38) + 10^2(0.03) - 0.16^2 \approx 4.4$

$\sigma_X = \sqrt{Var(X)} = \sqrt{4.4} \approx 2.1$.

Example 9 Find the variance and standard deviation of X with probability distribution given by

$$\Pr(X = x) = {}^3C_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x} \text{ for } x = 0,1,2,3 .$$

Set up the probability distribution table for X.

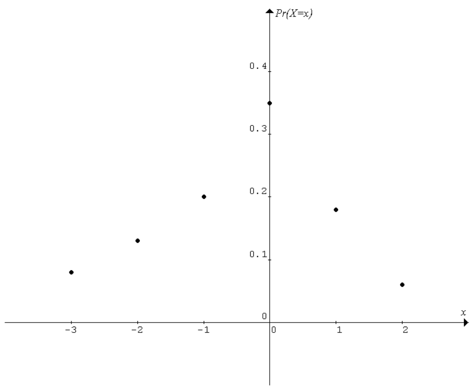
x	0	1	2	3
$\Pr(X = x)$	27/64	27/64	9/64	1/64

$$\mu_X = 0\left(\frac{27}{64}\right)+1\left(\frac{27}{64}\right)+2\left(\frac{9}{64}\right)+3\left(\frac{1}{64}\right) = 0.75$$

$$Var(X) = 0^2\left(\frac{27}{64}\right)+1^2\left(\frac{27}{64}\right)+2^2\left(\frac{9}{64}\right)+3^2\left(\frac{1}{64}\right)-0.75^2 = 0.5625$$

$$\sigma_X = \sqrt{0.5625} = 0.75$$

Example 10 Calculate the mean, variance and standard deviation of the following probability distribution.
Find $\Pr(\mu-\sigma \le X \le \mu+\sigma)$ and $\Pr(\mu-2\sigma \le X \le \mu+2\sigma)$.



Set up the probability distribution table for X.

x	-3	-2	-1	0	1	2
$\Pr(X = x)$	0.08	0.13	0.2	0.35	0.18	0.06

$$\mu_X = -3(0.08)+(-2)(0.13)+(-1)(0.2)+0(0.35)+1(0.18)+2(0.06) = -0.4$$

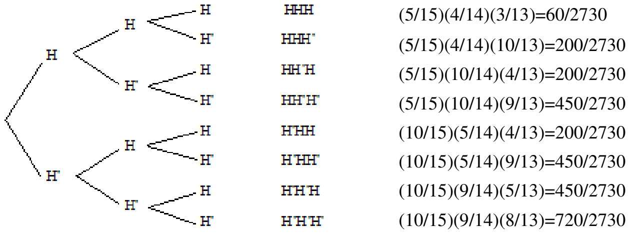
$$Var(X) = (-3)^2(0.08)+(-2)^2(0.13)+(-1)^2(0.2)+0^2(0.35)+1^2(0.18)+2^2(0.06)-(-0.4)^2$$
$$= 1.7 .$$

$$\sigma_X = \sqrt{1.7} \approx 1.3$$

$$\Pr(\mu-\sigma \le X \le \mu+\sigma) = \Pr(-1.7 \le X \le 0.9) = 0.2+0.35 = 0.55$$

$$\Pr(\mu-2\sigma \le X \le \mu+2\sigma) = \Pr(-3.0 \le X \le 2.2)$$
$$= 0.08+0.13+0.2+0.35+0.18+0.06 = 1 .$$

Example 11 Out of 15 pairs of socks 5 pairs have holes in them. You randomly select 3 pairs for an interstate trip. (a) Draw a tree diagram to show the possible selections. (b) Tabulate the probability distribution of random variable X-number of pairs with holes. (c) Find $\Pr(X<2)$ (d) Find $E(X)$ (e) Find $Var(X)$ (f) Find $\Pr(\mu-2\sigma \le X \le \mu+2\sigma)$.



(b)

x	0	1	2	3
$\Pr(X = x)$	720/2730	1350/2730	600/2730	60/2730

(c) $\Pr(X<2) = \Pr(X=0)+\Pr(X=1) = 69/91$

(d) $E(X) = 0(720/2730)+1(1350/2730)+2(600/2730)$
$+3(60/2730) = 1$

(e) $Var(X) = 0^2(720/2730)+1^2(1350/2730)+2^2(600/2730)$
$+3^2(60/2730)-1^2 = 4/7$

(f) $\sigma = \sqrt{Var(X)} = \sqrt{4/7} \approx 0.7559$

$\Pr(\mu-2\sigma \le X \le \mu+2\sigma)$
$= \Pr(1-2\times0.7559 \le X \le 1+2\times0.7559)$
$= \Pr(-0.5118 \le X \le 2.5118) = \Pr(0 \le X \le 2) \approx 0.98$

Example 12 Show that $Var(X) = E(X^2)-(\mu_X)^2$.

By definition, $Var(X) = E((X-\mu_X)^2)$

$\therefore\ Var(X) = E(X^2-2\mu_X X+\mu_X{}^2) = E(X^2)-E(2\mu_X X)+E(\mu_X{}^2)$
$= E(X^2)-2\mu_X E(X)+E(\mu_X{}^2)$
$= E(X^2)-2\mu_X{}^2+\mu_X{}^2 = E(X^2)-\mu_X{}^2$

Expected value and variance of random variable $Y = aX+b$

$E(Y) = E(aX+b) = aE(X)+b$

$Var(Y) = Var(aX+b) = a^2Var(X)$

$\sigma_Y = \sigma_{aX+b} = a\sigma_X$

Example 13 Find the expected value, variance and standard deviation of $Y = 2X-3$, where X is as defined in example 1.

$E(X) = 0.16$, $Var(X) = 4.4$, $\sigma_X = 2.1$.

$\therefore E(Y) = 2E(X)-3 = 2\times0.16-3 = -2.68$

$Var(Y) = 2^2Var(X) = 4\times4.4 = 17.6$

$\sigma_Y = 2\sigma_X = 2\times2.1 = 4.2$ or $\sigma_Y = \sqrt{Var(Y)} = \sqrt{17.6} = 4.2$

Example 14 The probability distribution of the weekly sales, X, of a particular brand of iPod has $\mu = 5.7$ and $\sigma = 2.8$. If the fixed cost in selling this brand of iPod is \$35 per week and the profit on each unit sold is \$38, write down the random variable Y, the weekly net profit, in terms of X. Hence find the expected weekly net profit and the standard deviation of the weekly net profit. Find the range of weekly net profit that the shopkeeper expects to get with 0.68 probability.

$Y = 38X-35$

$E(Y) = 38E(X)-35 = 38\times5.7-35 = 181.60$

$\sigma_Y = 38\sigma_X = 38\times2.8 = 106.40$

The shopkeeper expects $\Pr(\mu-\sigma \le Y \le \mu+\sigma) \approx 0.68$,
$(181.60-106.40) \le Y \le (181.60+106.40)$, $\therefore 75.20 \le Y \le 288$.

Example 15 Find p, q and r in the following probability distribution of X such that $\mu_X = 2$ and $Var(X) = 1$.

x	0	1	2	3
$\Pr(X = x)$	p	q	r	0.4

$\sum \Pr(X = x) = 1$

$\therefore\ p+q+r+0.4 = 1$ $\therefore p+q+r = 0.6$ ………………… (1)

$\mu_X = 2$

$\therefore\ 0p+1q+2r+3\times0.4 = 2$ $\therefore\ q+2r = 0.8$ …………… (2)

$Var(X) = 1$

$\therefore\ 0^2p+1^2q+2^2r+3^2\times0.4-2^2 = 1$ $\therefore q+4r = 1.4$ …… (3)

Solve equations (1), (2) and (3) simultaneously to obtain $p = 0.1$, $q = 0.2$ and $r = 0.3$.

Questions: Next page

1. Find the expectation of the following distribution.

x	-3	-1	1	5
$\Pr(X=x)$	0.2	0.3	0.4	0.1

2. Find the mean, variance and standard deviation of X with probability distribution given by

$$\Pr(X=x)={}^2C_x\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{2-x} \text{ for } x=0,\ 1,\ 2.$$

3. A casino allows a gambler to toss a fair die and to receive as many dollars as the number of dots that appear uppermost. The gambler pays $4 to play each game. Calculate the expected gain of the casino for each game.

4. Determine the median and the mode of the following probability distribution of X.

x	1.2	1.4	1.5	1.8	2
$\Pr(X=x)$	0.01	0.12	0.3	0.22	0.35

5. Determine the median and the mode of the following probability distribution of X.

x	1	1.2	1.5	1.8	2.3
$\Pr(X=x)$	0.03	0.47	0.05	0.2	0.25

6. Calculate the variance and standard deviation of the following probability distribution of X.

x	-2	1	4	7
$\Pr(X=x)$	0.2	0.3	0.4	0.1

7. For the probability distribution of X in Q6, find $\Pr(\mu-2\sigma\le X\le\mu+2\sigma)$.

8. Find the expected value, variance and standard deviation of $Y=3X-1$, where X is as defined in Q1.

9. Find a, b and c in the following probability distribution of X such that $\mu_X=1$ and $Var(X)=1.4$.

x	-1	0	1	2
$\Pr(X=x)$	a	b	c	0.5

10. For scores X on a nationwide competition, $\mu=120$ and $Var(X)=100$. Find the mean and variance of (a) $2X-15$ and (b) $\dfrac{X-120}{10}$.

11. A student buys a raffle ticket for $2. For every 1000 tickets sold, 2 mountain bikes are to be given away in a drawing.
(a) What is the probability of winning a bike?
(b) If each bike is worth $320, determine the student's expected gain.

Numerical, algebraic and worded answers.

1. 0 2. 2/3, 4/9, 2/3
3. $3 4. 1.8, 2
5. 1.35, 1.2 6. 7.56, 2.75
7. 1 8. -1, 45, 6.71
9. $a=0.2$, $b=0.1$, $c=0.2$
10. (a) 225, 400 (b) 0, 1
11. (a) 1/500 (b) $-$\$1.36