Free maths self-help revision

Graphs of transformed functions

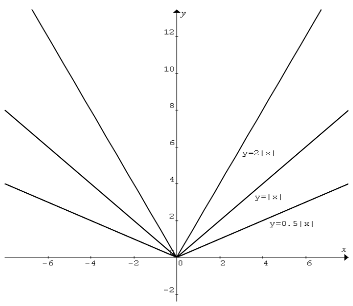
All functions or relations can be changed by one or a combination of the following transformations.

Vertical dilation of function $y = f(x)$:

$$y = f(x) \rightarrow \frac{y}{A} = f(x) \text{ where } A > 0 \text{, .: } y = Af(x)$$

For $0 \le A < 1$*, the graph of* $y = f(x)$ *is compressed towards the x-axis to give it a wider appearance; for* $A > 1$*, it is stretched away from the x-axis to give it a narrower appearance.*
A is called the dilation factor.
Other ways to say vertical dilation are:
- dilation parallel to the y-axis
- dilation from (or towards) the x-axis

Example 1 Compare the graphs of the transformed functions $y = \frac{1}{2}|x|$ and $y = 2|x|$ with the graph of the original function $y = |x|$.

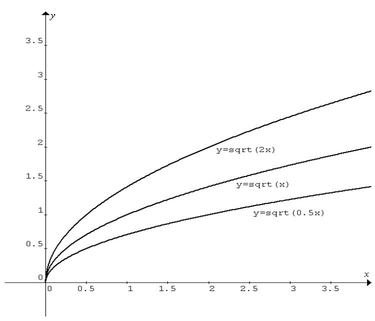


Horizontal dilation of function $y = f(x)$:

$$y = f(x) \rightarrow y = f\left(\frac{x}{n}\right) \text{ where } n > 0$$

For $0 < n < 1$*, the graph of* $y = f(x)$ *is compressed towards the y-axis to give it a narrower appearance; for* $n > 1$*, it is stretched away from the y-axis to give it a wider appearance.*
The dilation factor is n *for this transformation.*
Other ways to say horizontal dilation are:
- dilation parallel to the x-axis
- dilation from (or towards) the y-axis

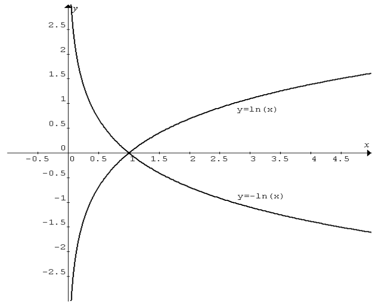
Example 2 Compare $y = \sqrt{0.5x}$ and $y = \sqrt{2x}$ with $y = \sqrt{x}$.



Reflection of function $y = f(x)$ *in the x-axis*:

$$y = f(x) \rightarrow -y = f(x), \text{ .: } y = -f(x)$$

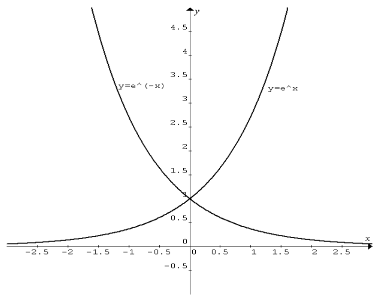
Example 3 Compare $y = -\log_e x$ with $y = \log_e x$.



Reflection of function $y = f(x)$ *in the y-axis*:

$$y = f(x) \rightarrow y = f(-x)$$

Example 4 Compare $y = e^{-x}$ with $y = e^{x}$.

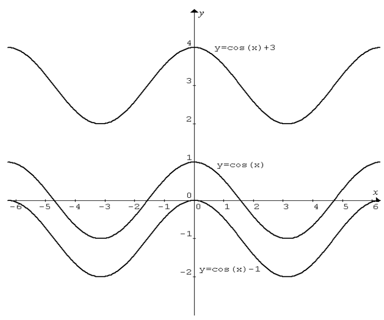


Vertical translation of function $y = f(x)$ *by c units*:

$$y = f(x) \rightarrow y \pm c = f(x) \text{ where } c > 0 \text{, .: } y = f(x) \mp c$$

The + and − operations correspond to downward and upward translations respectively in $y \pm c = f(x)$*, but upward and downward translations respectively in* $y = f(x) \mp c$.

Example 5 Compare $y = \cos x + 3$ and $y = \cos x - 1$ with $y = \cos x$.



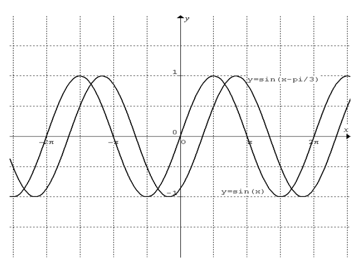
Horizontal translation of function $y = f(x)$ *by b units*:

$$y = f(x) \rightarrow y = f(x \pm b) \text{, where } b > 0$$

The + and − operations correspond to left and right translations respectively.

After translations (horizontal and/or vertical), the size and shape remain the same as the original.

Example 6 Compare $y = \sin\left(x - \frac{\pi}{3}\right)$ and $y = \sin x$.



Sequence of transformations

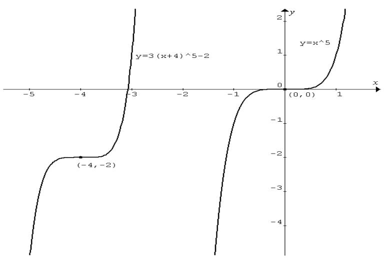
If a transformed function is the result of a sequence of the above transformations, it would be easier to recognise what the transformations are by expressing the function in the form

$$y = \pm Af\left(\pm\frac{1}{n}(x \pm b)\right) \pm c$$

vertical translation (+ up)
horizontal translation (+ left)
horizontal dilation (factor n)
reflection in the y-axis (− sign)
vertical dilation (factor A)
reflection in the x-axis (− sign)

To sketch the transformed function from the original function, always carry out the translations last when the transformed function is written in the above form.

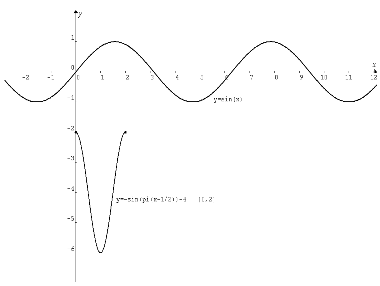
Example 7 Sketch $y = 3(x+4)^5 - 2$.
This function is the result of a combination of transformations of $y = x^5$. It involves a vertical dilation by a factor of 3, and translations of 4 left and 2 down. The stationary point of inflection changes from $(0,0)$ to $(-4,-2)$.



Example 8 Sketch $y = -2\sin\left(\pi x - \frac{\pi}{2}\right) - 4$ for $0 \le x \le 2$.

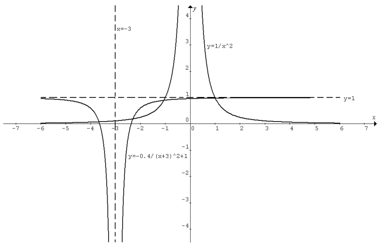
Express the function as $y = -2\sin\pi\left(x - \frac{1}{2}\right) - 4$.

This function is the result of a sequence of transformations of $y = \sin x$. The amplitude changes from 1 to 2 (note: not –2) and the period changes from 2π to $T = \frac{2\pi}{\pi} = 2$. They correspond to a vertical dilation of the function by a factor of 2 and a horizontal dilation by a factor of $\frac{1}{\pi}$ respectively. There is a reflection in the *x*-axis followed by translations of $\frac{1}{2}$ right and 4 down.



Example 9 Sketch $y = \frac{-0.4}{(x+3)^2} + 1$.

This function is the result of a sequence of transformations of $y = \frac{1}{x^2}$. It involves a reflection in the *x*-axis and a vertical dilation by a factor of 0.4, and then translations 3 left and 1 up. The function has $x = -3$ and $y = 1$ as its asymptotes.



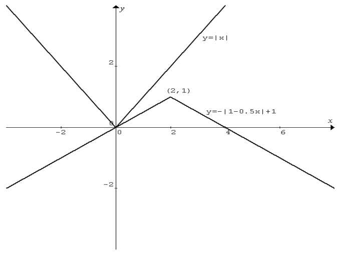
Example 10 Sketch $y = -\left|1 - \frac{x}{2}\right| + 1$.

It is the result of a sequence of transformations of $y = |x|$.

Firstly express it in the form $y = \pm Af\left(\pm\frac{1}{n}(x \pm b)\right) \pm c$.

$y = -\frac{1}{2}|2 - x| + 1 = -\frac{1}{2}|-(x-2)| + 1 = -\frac{1}{2}|x-2| + 1$.

The sequence of transformations is:
Reflection in the *x*-axis; vertical dilation by a factor of $\frac{1}{2}$; translations 2 right and 1 up. The vertex is $(2,1)$.



Implicit form

A function or relation can be written in implicit form, e.g. $\frac{1}{2}(x-1)^2 - 3\sqrt{-y+2} = 3$.
The general rules for transformations are:

(1) Change x to $-x$ *for reflection in the y-axis.*
(2) Change y to $-y$ *for reflection in the x-axis.*
(3) Change x to $\frac{x}{a}$ *for dilation from the y-axis by a factor of a.*
(4) Change y to $\frac{y}{a}$ *for dilation from the x-axis by a factor of a.*
(5) Change x to $x - a$ *for horizontal translation by +a units.*
(6) Change y to $y - a$ *for vertical translation by +a units.*

These transformations can be carried out in **any order.*

Example 11 The graph of $y = \frac{1}{1-x}$ undergoes the following transformations: Translate upwards by 2 units, reflect in the *x*-axis and then dilate vertically by a factor of 0.5. Write down the equation of the transformed function.

$$y = \frac{1}{1-x} \to y - 2 = \frac{1}{1-x} \to -y - 2 = \frac{1}{1-x} \to -2y - 2 = \frac{1}{1-x}$$

$\therefore\ y = \frac{1}{2(x-1)} - 1$

Example 12 Find the equation of the transformed function when $y = 1 - \sqrt{2-x}$ undergoes the sequence of transformations:
(1) downward translation by 2 units, (2) horizontal dilation by a factor of 2 and (3) reflection in the *y*-axis.

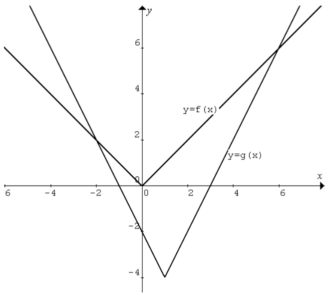
$$y = 1 - \sqrt{2-x} \to y + 2 = 1 - \sqrt{2-x} \to y + 2 = 1 - \sqrt{2 - \frac{x}{2}}$$

$\to y + 2 = 1 - \sqrt{2 - \frac{-x}{2}}$, i.e. $\to y = -1 - \sqrt{2 + \frac{x}{2}}$

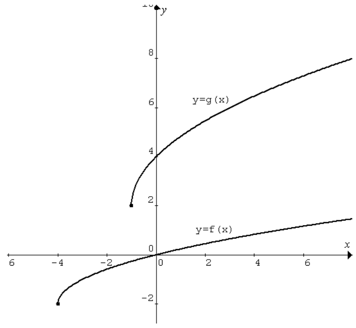
Questions: Next page

Q1 $y=f(x)$ is transformed to $y=g(x)$. State the sequence of transformations in each case.

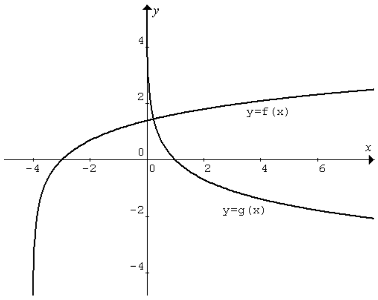
a



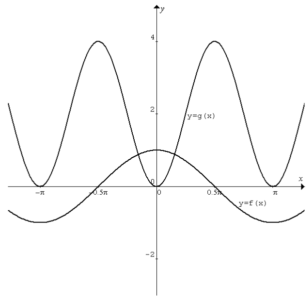
Q1b



Q1c



Q1d



Q2 Starting from $y=f(x)$, sketch the graph $y=g(x)$.

a $f(x)=2e^x-2$, $g(x)=\frac{1}{2}(f(x)+2)$

Q2b $f(x)=\sin\left(\frac{x}{2}\right)$, $g(x)=2f(x+\pi)-2$

Q3 Sketch $y-1=\frac{1}{x-1}$ by transformations. Clearly show the asymptotes.

Q4 Sketch $y=2-\frac{1}{(x+2)^2}$ by transformations. Clearly show the asymptotes.

Q5 $y=2|1+x|-1$ undergoes the following transformations in the order shown: (1) Translate to the left by 1 unit (2) Reflect in the y-axis (3) Dilate horizontally by a factor of 0.5. Find the equation of the transformed function.

Q6 $y^2=2\left(x-\frac{1}{2}\right)$ undergoes the following transformations in the order shown: (1) Translate to the right by 1 unit (2) Reflect in the x-axis (3) Dilate vertically by a factor of 0.5. Find the equation of the transformed function.

Q7 $\sqrt{y+1}=1-\frac{x}{2}$ undergoes the following transformations in the order shown: (1) Dilate horizontally by a factor of 0.5. (2) Reflect in the y-axis (3) Translate to the right by 1 unit (4) Translate upward by 1 unit. Find the equation and sketch the graph of the transformed function.

Numerical, algebraic and worded answers:

1a. Vert. dil. by factor 2, right 1 down 4.

1b. Vert. dil. by factor 2, right 3 up 4. 1c. Reflected in the x-axis, right 4.

1d. Reflected in the x-axis, hori. dil. by factor ½, vert. dil. by factor 2, up 2.

3. Asymptotes: $x=1$, $y=1$

4. Asymptotes: $x=-2$, $y=2$

5. $y=4|1-x|-1$

6. $y^2=\frac{1}{2}\left(x-\frac{3}{2}\right)$

7. $x=\sqrt{y}$, the right half of parabola $y=x^2$