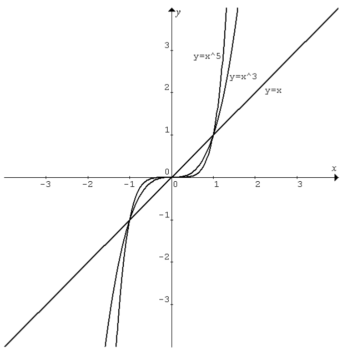
itute.com

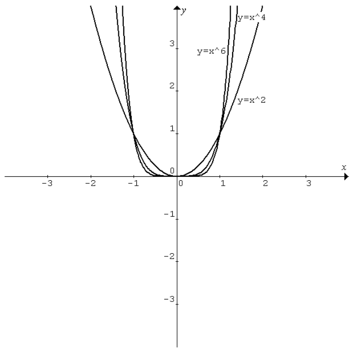
Free maths self-help revision © copyright itute 2013

Power functions *$y = x^n$, for $n \in Q$ (set of rational numbers)*

The graph of $y = x$ is a straight line through the origin $(0,0)$.
Domain: R; range: R.
The graphs of odd-power functions, e.g. $y = x^3$ and $y = x^5$ have a stationary point of inflection at $(0,0)$.
Domain: R; range: R.

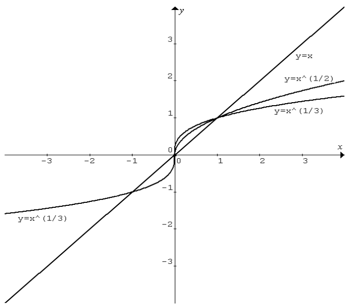


The graphs of even-power functions, e.g. $y = x^2$ and $y = x^4$ have a turning-point at $(0,0)$.
They show symmetry under a reflection in the y-axis. ∴the y-axis, i.e. the line $x = 0$ is called the axis of symmetry.
Domain: R; range: $[0, \infty)$.



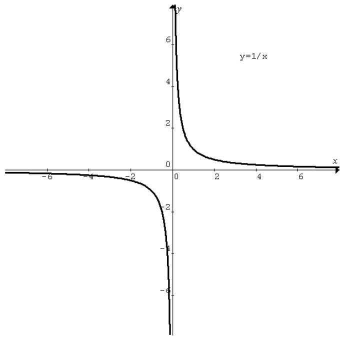
The function $y = x^{\frac{1}{2}}$ can be written as $y = \sqrt{x}$.
It is undefined for $x < 0$. It has an end point at $(0,0)$.
Domain: $[0, \infty)$; range: $[0, \infty)$.

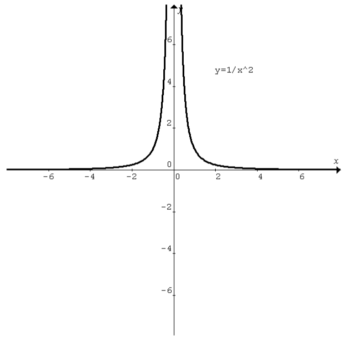
The function $y = x^{\frac{1}{3}}$ can be expressed as $y = \sqrt[3]{x}$.
It has a vertical tangent at $(0,0)$. Domain: R; range: R.



The graph of $y = x^{-1}$ (or $y = \dfrac{1}{x}$) consists of two branches, one in the first quadrant and the other in the third quadrant. The axes of symmetry are lines $y = \pm x$.
The function shows the following asymptotic behaviours:
As $x \to -\infty$, $y \to 0^-$; as $x \to +\infty$, $y \to 0^+$.
$\therefore y = 0$ is the horizontal asymptote of the function.
As $x \to 0^-$, $y \to -\infty$; as $x \to 0^+$, $y \to +\infty$.
$\therefore x = 0$ is the vertical asymptote.
Domain: R\{0}; range: R\{0}.

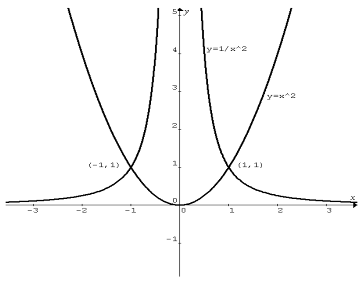
The graph of $y = x^{-2}$ (or $y = \dfrac{1}{x^2}$) also has two branches, one in the first quadrant and the other in the second quadrant. The line $x = 0$ is the axis of symmetry.
The function shows the following asymptotic behaviours:
As $x \to -\infty$, $y \to 0^+$; as $x \to +\infty$, $y \to 0^+$.
$\therefore y = 0$ is the horizontal asymptote of the function.
As $x \to 0^-$, $y \to +\infty$; as $x \to 0^+$, $y \to +\infty$.
$\therefore x = 0$ is the vertical asymptote.
Domain: R\{0}; range: R^+.





Example 1 Find the coordinates of the intersection(s) of $y = \dfrac{1}{x^2}$ and $y = x^2$ algebraically.

Substitution: $x^2 = \dfrac{1}{x^2}$ for $x \neq 0$, .: $x^4 = 1$, $x = \pm 1$ and $y = x^2 = 1$.
The intersections are $(-1,1)$ and $(1,1)$.

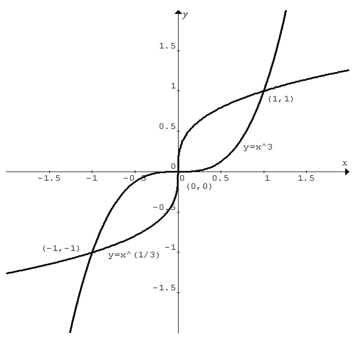


Example 2 Find the coordinates of the intersection(s) of $y = x^{\frac{1}{3}}$ and $y = x^3$ algebraically.

Substitution: $x^3 = x^{\frac{1}{3}}$.

Take the cube of both sides: $\left(x^3\right)^3 = \left(x^{\frac{1}{3}}\right)^3$, .: $x^9 = x$,

$x^9 - x = 0$, $x\left(x^8 - 1\right) = 0$, .: $x = 0$ or ± 1 and the corresponding y-coordinates are 0, ± 1.
The intersections are $(-1,-1)$, $(0,0)$ and $(1,1)$.

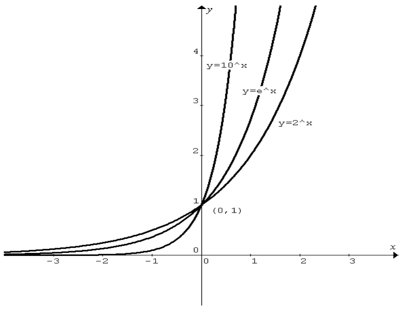


Exponential functions $y = a^x$ *where* $a \in R^+$

For $a > 1$*, the graphs of functions with equation* $y = a^x$ *have the same shape and the same y-intercept* $(0,1)$.

Asymptotic behaviour: As $x \to -\infty$, $y \to 0^+$*, the same horizontal asymptote* $y = 0$ *for the functions.*

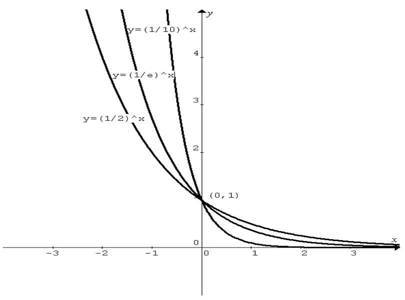
a^x *is always > 0. Domain: R and range:* R^+.



For $0 < a < 1$*, the graphs of* $y = a^x$ *have the same shape and the same y-intercept* $(0,1)$.

Asymptotic behaviour: As $x \to +\infty$, $y \to 0^+$*, the same horizontal asymptote* $y = 0$ *for the functions.*

a^x *is always > 0. Domain: R and range:* R^+.



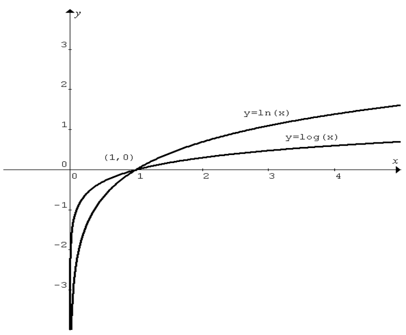
Example 3 Show that the graphs of $y = a^x$ and $y = \left(\frac{1}{a}\right)^x$ for $a \in R^+$ are reflections of each other in the *y*-axis.

$y = \left(\frac{1}{a}\right)^x = \left(a^{-1}\right)^x = a^{-x}$, the negative sign before *x* means $y = a^{-x}$ is the reflection of $y = a^x$ in the *y*-axis and vice versa.

Logarithmic functions $y = \log_a x$ *where* $a \in R^+$

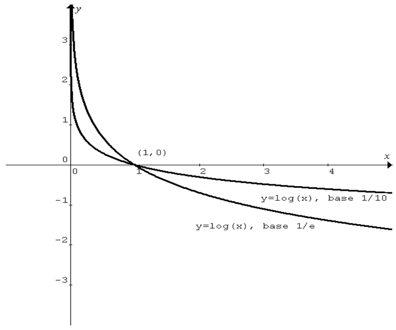
For $a > 1$*, e.g. the logarithmic functions* $y = \log_e x$ *(* $\ln(x)$ *on your calculators) and* $y = \log_{10} x$ *(* $\log(x)$ *on calculators) have the same shape and a common x-intercept* $(1,0)$*. They are undefined for* $x \le 0$*. Both functions have a negative value for* $0 < x < 1$ *and a positive value for* $x > 1$.

Asymptotic behaviour: As $x \to 0^+$, $y \to -\infty$*, the same vertical asymptote* $x = 0$*. Domain:* R^+ *and range: R.*



For $0 < a < 1$*, e.g.* $y = \log_{\frac{1}{e}} x$ *and* $y = \log_{\frac{1}{10}} x$ *have the same shape and a common x-intercept* $(1,0)$*. They are undefined for* $x \le 0$*. Both functions have a positive value for* $0 < x < 1$ *and a negative value for* $x > 1$.

Asymptotic behaviour: As $x \to 0^+$, $y \to \infty$*, the same vertical asymptote* $x = 0$*. Domain:* R^+ *and range: R.*



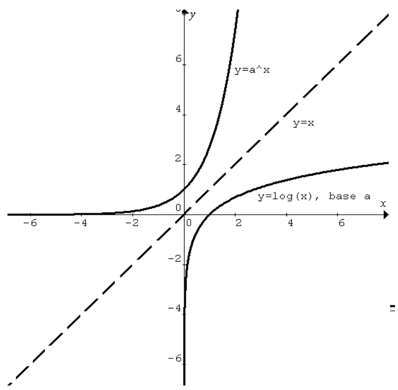
Example 4 Show that the graphs of $y = \log_{\frac{1}{a}} x$ and $y = \log_a x$ are reflections of each other in the *x*-axis for $a \in R^+$.

$y = \log_{\frac{1}{a}} x$, .: $x = \left(\frac{1}{a}\right)^y = \left(a^{-1}\right)^y = a^{-y}$, .: $-y = \log_a x$

The negative sign before *y* means $y = \log_{\frac{1}{a}} x$ is the reflection of $y = \log_a x$ in the *x*-axis and vice versa.

Example 5 What do you notice about the graphs of $y = a^x$ and $y = \log_a x$ for $a \in R^+$?

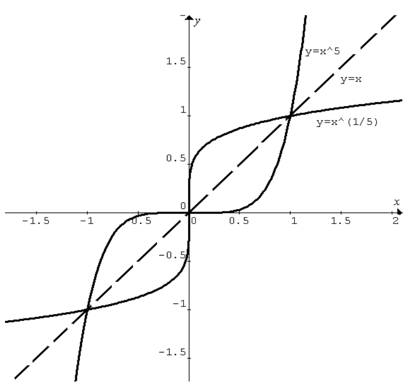
Sketch the two graphs on the same axes of *the **same scale***.



They are reflections of each other in the line $y = x$.

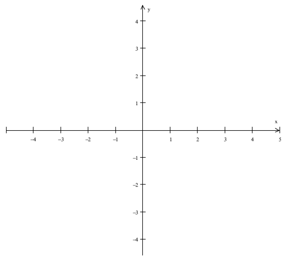
Any *two functions with this property are **inverses** of each other.*

Example 6 Are $y = x^5$ and $y = x^{\frac{1}{5}}$ inverses of each other?

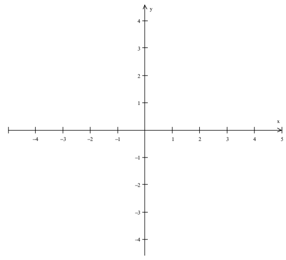


Questions: Next page

Q1 Sketch the graphs of x^2, x^4 and x^6 on the same set of axes.



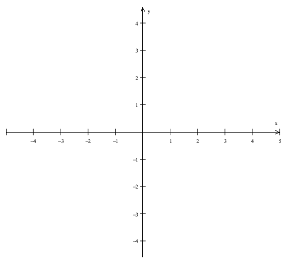
Q2 Sketch the graphs of x^n for $n = 1$, 3 and 5 on the same set of axes.



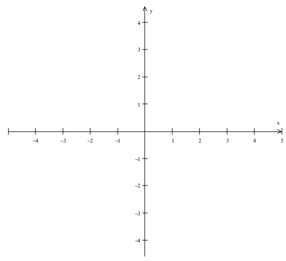
Q3 Refer to the even-power functions in Q1 and the odd-power functions in Q2. Which set of power functions shows symmetry under reflection in the *y*-axis?

Q4 Find the coordinates of the intersection(s) of $y = x^4$ and $y = x^{\frac{1}{4}}$.

Q5 Sketch the graphs of the inverse of x^n for $n = 2$ and 4 on the same set of axes.

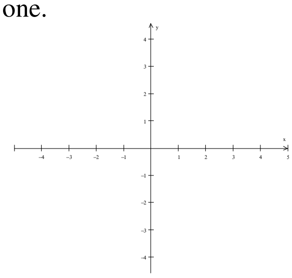


Q6 Sketch the graphs of the inverse of x^n for $n = 3$ and 5 on the same set of axes.



Q7 Refer to Q5. Find the equation and domain of each inverse.

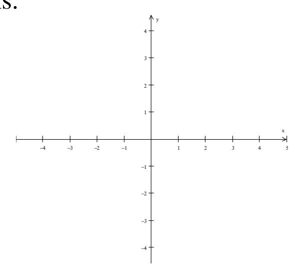
Q8 Sketch the graphs of 2^x and 5^x on the same set of axes. State the domain and range of each one.



Q9 Write down the functions which are the reflections of 2^x and 5^x in the line $y = x$.

Q10 Write down the functions which are the reflections of 2^x and 5^x in the *y*-axis.

Q11 Sketch the graphs of $\log_2 x$ and $\log_{\frac{1}{2}} x$ on the same set of axes. Comment on the two graphs.



Numerical, algebraic and worded answers:

3. Even-power functions
4. $(0,0)$, $(1,1)$
7. $y = x^{\frac{1}{2}}$, domain $[0,\infty)$; $y = x^{\frac{1}{4}}$, domain $[0,\infty)$
8. Both have the same domain *R*, range $(0,\infty)$
9. $\log_2 x$, $\log_5 x$
10. 2^{-x}, 5^{-x}
11. They are reflections of each other in the *x*-axis.