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Applications of functions and calculus

Curve sketching

Sketching the graph of a function requires the knowledge of the x and y-intercepts, the asymptotic behaviours and the position and nature of each stationary point.

The x-coordinate of a stationary point can be found by letting $\frac{dy}{dx}=0$*, or* $f'(x)=0$*, then solve for x. The y-coordinate is found by substituting the x value in the equation of the function.*

The nature of a stationary point can be determined by finding the value of $\frac{dy}{dx}$ *on each side of the stationary point.*

Left	**Stationary point**	**Right**	**Nature**
$\frac{dy}{dx}>0$	$\frac{dy}{dx}=0$	$\frac{dy}{dx}<0$	Local max.
$\frac{dy}{dx}<0$	$\frac{dy}{dx}=0$	$\frac{dy}{dx}>0$	Local min.
$\frac{dy}{dx}>0$	$\frac{dy}{dx}=0$	$\frac{dy}{dx}>0$	Inflection pt.
$\frac{dy}{dx}<0$	$\frac{dy}{dx}=0$	$\frac{dy}{dx}<0$	Inflection pt.

Note: A local maximum (local minimum) is not necessarily the maximum (minimum) point of the function. You need to compare it with other stationary points and end points in the domain.

Example 1 Sketch $y=x^3-3x$, $x\in[-2,2]$.

End points: $x=-2,\ y=-2$; $x=2,\ =y=2$.

x-intercepts: Let $y=0,\ x^3-3x=0,\ x(x^2-3)=0,$
$x(x-\sqrt{3})(x+\sqrt{3})=0,\ x=-\sqrt{3},0,\sqrt{3}$.

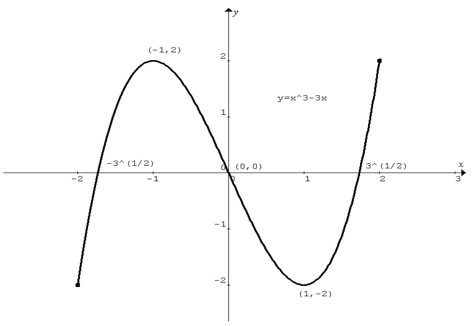
Stationary points: Let $\frac{dy}{dx}=0,\ \frac{dy}{dx}=3(x^2-1)=0,$.: $x=-1,\ 1$

.: $y=2,\ -2$

Nature of stationary points:

At $x=$	-2	-1	0	1	2
$\frac{dy}{dx}$	>0	$=0$	<0	$=0$	>0
Nature		Local max.		Local min.	

Asymptotic behaviour: None.



Example 2 Sketch the graph of $x=\frac{1}{10}(t-5)^2e^{-0.1t}$, $t\in[0,\infty)$.

t-intercepts: Let $x=0,\ \frac{1}{10}(t-5)^2e^{-0.1t}=0.$

Since $e^{-0.1t}>0$, .: $t-5=0$ or $t=5$. This intercept is also a turning point as indicated by $(t-5)$ being a repeated factor.

x-intercept: Let $t=0,\ x=2.5$. This is also an end point.

Asymptotic behaviour:

As $t\to\infty$, the dominant factor $e^{-0.1t}\to 0^+$, .: $x\to 0^+$.

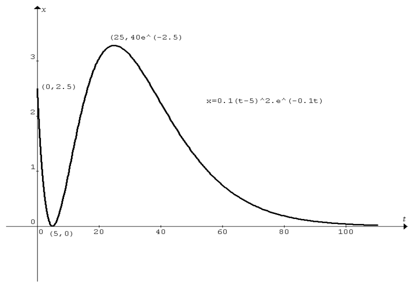
Stationary points: $\frac{dx}{dt}=0,\ -0.01(t-5)^2e^{-0.1t}+0.2(t-5)e^{-0.1t}=0,$

$-(t-5)e^{-0.1t}(0.01(t-5)-0.2)=0,\ -(t-5)(0.01t-0.25)e^{-0.1t}=0.$

Since $e^{-0.1t}>0$, .: $t-5=0$, i.e. $t=5$ and $x=0$ (as discussed earlier), or $0.01t-0.25=0$, i.e. $t=25$ and $x=40e^{-2.5}$.

Nature of stationary points:

At $t=$	4	5	10	25	26
$\frac{dx}{dt}$	<0	$=0$	>0	$=0$	<0
Nature		Local min.		Local max.	



Example 3 Sketch $y=e^{\frac{x}{\sqrt{3}}}\cos(x)$ for $x\in[0,2\pi]$.

End points: At $x=0,\ y=1$. At $x=2\pi,\ y=e^{\frac{2\pi}{\sqrt{3}}}$.

x-intercepts: Let $y=0$, $e^{\frac{x}{\sqrt{3}}}\cos(x)=0$, and since $e^{\frac{x}{\sqrt{3}}}>0$,

.: $\cos(x)=0$, and hence $x=\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Asymptotic behaviour: None.

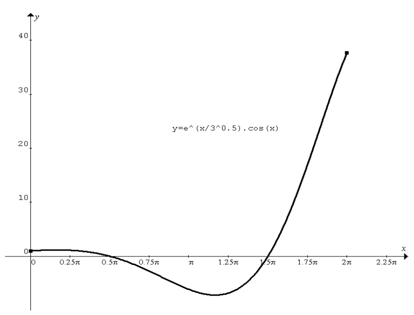
Stationary points: $\frac{dy}{dx}=0,\ -e^{\frac{x}{\sqrt{3}}}\sin(x)+\frac{1}{\sqrt{3}}e^{\frac{x}{\sqrt{3}}}\cos(x)=0,$

$-e^{\frac{x}{\sqrt{3}}}\left(\sin(x)-\frac{1}{\sqrt{3}}\cos(x)\right)=0,\ \sin(x)=\frac{1}{\sqrt{3}}\cos(x),\ \tan(x)=\frac{1}{\sqrt{3}}$ H

ence $x=\frac{\pi}{6}$ and $y=\frac{\sqrt{3}}{2}e^{\frac{\pi}{6\sqrt{3}}}$ or $x=\frac{7\pi}{6}$ and $y=-\frac{\sqrt{3}}{2}e^{\frac{7\pi}{6\sqrt{3}}}$.

Nature of stationary points:

At $x=$	0.5	$\frac{\pi}{6}$	1	$\frac{7\pi}{6}$	4
$\frac{dy}{dx}$	>0	$=0$	<0	$=0$	>0
Nature		Local max.		Local min.	



Example 4 Sketch $y=\frac{1}{x+2}+\frac{1}{3-x}+1$ for $x\in[-1,3)$.

End point: At $x=-1$, $y=2.25$.

Asymptotic behaviour: As $x\to 3^-$, $y\to\infty$.

The function is positive for $x\in[-1,3)$, no x-intercepts.

y-intercept: Let $x=0$, $y=\frac{11}{6}$.

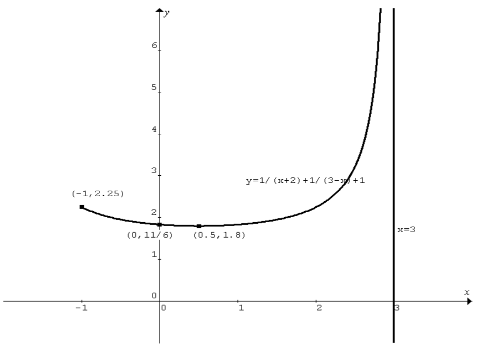
Stationary points: $\frac{dy}{dx}=-\frac{1}{(x+2)^2}+\frac{1}{(3-x)^2}=0$,

$\frac{(x+2)^2-(3-x)^2}{(3-x)^2(x+2)^2}=0$, $\therefore (x+2)^2-(3-x)^2=0$,

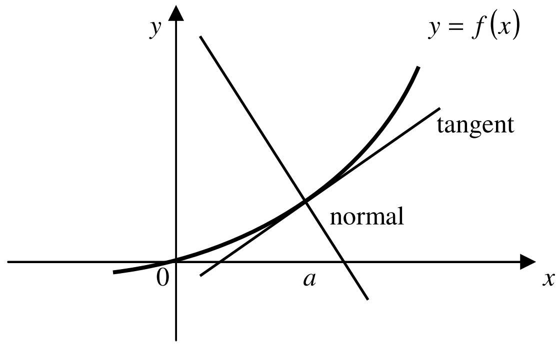
$((x+2)-(3-x))((x+2)+(3-x))=0$, $5(2x-1)=0$, $x=\frac{1}{2}$ and $y=\frac{9}{5}$.

Nature of stationary point:

At $x=$	0	$\frac{1}{2}$	1
$\frac{dy}{dx}$	<0	$=0$	>0
Nature		Local min.	



Equations of tangents and normals



Gradient of the tangent to the curve $y=f(x)$ *at* $x=a$ *is* $m_T=f'(a)$.

Gradient of the normal is $m_N=-\frac{1}{m_T}=-\frac{1}{f'(a)}$.

Use $y-y_1=m(x-x_1)$ *to find equations of tangents and normals.*

For tangents: $y-f(a)=f'(a)\times(x-a)$.

For normals: $y-f(a)=-\frac{1}{f'(a)}\times(x-a)$.

Example 5 Find the equation of the normal to the curve $y=3\log_e(2x+1)-1$ at $x=0$.

At $x=0$, $y=-1$, $m_T=\frac{dy}{dx}=\frac{6}{2x+1}=6$ and .: $m_N=-\frac{1}{6}$.

Equation of the normal: $y-^-1=-\frac{1}{6}(x-0)$, .: $y=-\frac{1}{6}x-1$.

Example 6 Find the equation of the tangent to the curve $y=a(x+1)(x-1)^2$ at the y-intercept.
Find the point where the tangent crosses the curve.
Explain why this point is independent of a.

At the y-intercept, $x=0$, $y=a$, $\frac{dy}{dx}=2a(x+1)(x-1)+a(x-1)^2$
$=a(x-1)(2(x+1)+x-1)=a(x-1)(3x+1)=-a$.

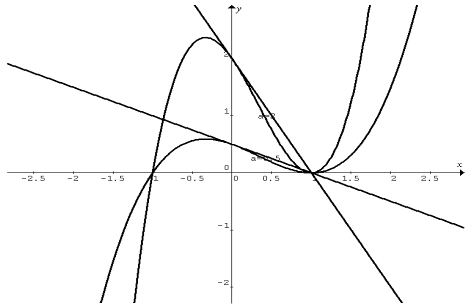
Equation of the tangent at the y-intercept: $y-a=-a(x-0)$, i.e. $y=-a(x-1)$.

Solve $y=-a(x-1)$ and $y=a(x+1)(x-1)^2$ simultaneously to find the intersection. $-a(x-1)=a(x+1)(x-1)^2$,

.: $a(x-1)+a(x+1)(x-1)^2=0$, $a(x-1)(1+(x+1)(x-1))=0$,

.: $a(x-1)x^2=0$, .: $x=0$ (where the line touches the curve) or $x=1$ (where the line crosses the curve) and $y=0$.

The intersecting point is $(1,0)$, and since the parameter a does not appear in the coordinates, it is independent of a.

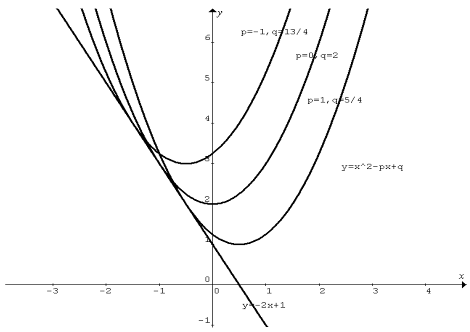


Example 7 The line $y=-2x+1$ is a tangent to the parabola $y=x^2-px+q$. Find the values of p and q.

$m_T=\frac{dy}{dx}=2x-p=-2$. .: $x=\frac{p-2}{2}$ is the x-coordinate of the point of contact. The y-coordinate is found by substituting $x=\frac{p-2}{2}$ in $y=-2x+1$, .: $y=3-p$,

or in $y=x^2-px+q$, .: $y=\left(\frac{p-2}{2}\right)^2-\frac{p(p-2)}{2}+q$.

.: $3-p=\left(\frac{p-2}{2}\right)^2-\frac{p(p-2)}{2}+q$, it can be simplified to $(2-p)^2=4(q-1)$. The values of p and q have to satisfy this relationship, e.g. if $p=0$, $q=2$; if $p=1$, $q=\frac{5}{4}$; if $p=-1$, $q=\frac{13}{4}$. There are infinitely many possibilities.



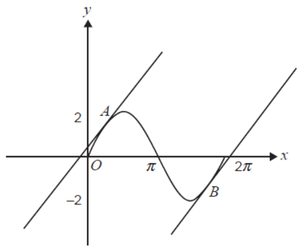
Exercise: Next page

Q1 (2006 VCAA Exam 1)

A **normal** to the graph of $y=\sqrt{x}$ has equation $y=-4x+a$, where a is a real constant. Find the value of a.

Q2 (2006 VCAA Exam 2)

Consider the function $f:[0, 2\pi] \to R$, $f(x) = 2\sin(x)$. The graph of f is shown below, with tangents drawn at points A and B.



a. **i.** Find $f'(x)$.

ii. Find the maximum and minimum values of $|f'(x)|$.

b. **i.** The gradient of the curve with equation $y=f(x)$, when $x=\frac{\pi}{3}$, is 1. Find the **other** value of x for which the gradient of the curve, with equation $y=f(x)$, is 1. (The exact value must be given.)

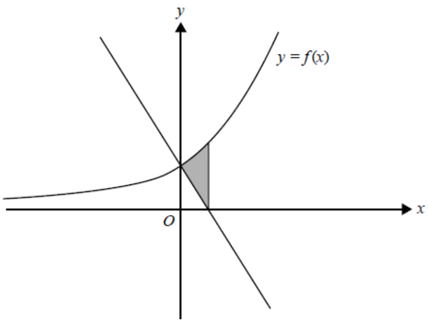
ii. Find the equation of the tangent to the curve at $x=\frac{\pi}{3}$. (Exact values must be given.)

iii. Find the axes intercepts of the tangent found in **b. ii.** (Exact values must be given.)

c. The two tangents to the curve at points A and B have gradient 1. A translation of m units in the positive direction of the x-axis takes the tangent at A to the tangent at B. Find the exact value of m.

Q3 (2007 VCAA Exam 1)

The graph of $f: R \to R$, $f(x) = e^{\frac{x}{2}} + 1$ is shown. The normal to the graph of f where it crosses the y-axis is also shown.



Find the equation of the **normal** to the graph of f where it crosses the y-axis.

Q4 (2008 VCAA Exam 2)

The graph of $y = x^3 - 12x$ has turning points where $x = 2$ and $x = -2$.

The graph of $y = |x^3 - 12x|$ has a positive gradient for

A. $x \in R$

B. $x \in \{x : x < -2\} \cup \{x : x > 2\}$

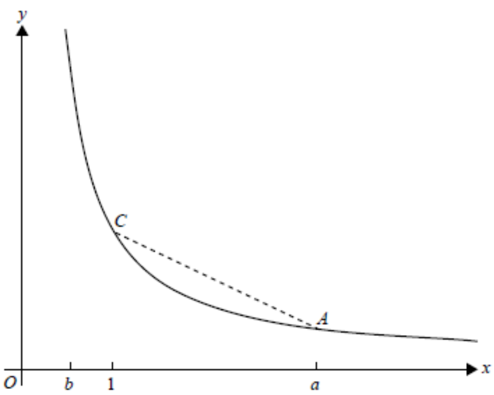
C. $x \in \{x : x < -2\sqrt{3}\} \cup \{x : x > 2\sqrt{3}\}$

D. $x \in \{x : -2\sqrt{3} < x < -2\} \cup \{x : 0 < x < 2\} \cup \{x : x > 2\sqrt{3}\}$

E. $x \in \{x : -2 < x < 0\} \cup \{x : 2 < x < 2\sqrt{3}\} \cup \{x : x > 2\sqrt{3}\}$

Q5 (2008 VCAA Exam 2)

The diagram below shows part of the graph of the function $f: R^+ \to R$, $f(x) = \frac{7}{x}$.



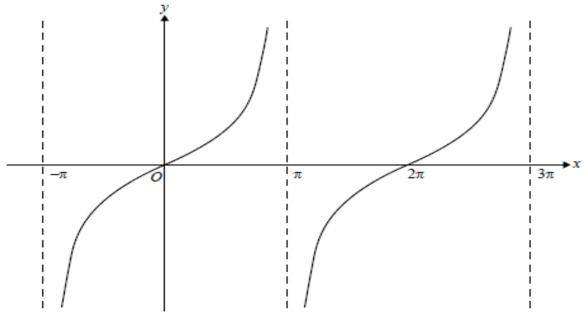
The line segment CA is drawn from the point $C(1, f(1))$ to the point $A(a, f(a))$ where $a > 1$.

a. **i.** Calculate the gradient of CA in terms of a.

ii. At what value of x between 1 and a does the tangent to the graph of f have the same gradient as CA?

Q6 (2008 VCAA Exam 2)

The graph of $f: (-\pi, \pi) \cup (\pi, 3\pi) \to R$, $f(x) = \tan\left(\frac{x}{2}\right)$ is shown below.



a. **i.** Find $f'\left(\frac{\pi}{2}\right)$.

ii. Find the equation of the **normal** to the graph of $y = f(x)$ at the point where $x = \frac{\pi}{2}$.

iii. Sketch the graph of this normal on the axes above. Give the exact axis intercepts.

b. Find the exact values of $x \in (-\pi, \pi) \cup (\pi, 3\pi)$ such that $f'(x) = f'\left(\frac{\pi}{2}\right)$.

Let $g(x) = f(x - a)$.

c. Find the exact value of $a \in (-1, 1)$ such that $g(1) = 1$.

Let $h: (-\pi, \pi) \cup (\pi, 3\pi) \to R$, $h(x) = \sin\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right) + 2$.

d. **i.** Find $h'(x)$.

ii. Solve the equation $h'(x) = 0$ for $x \in (-\pi, \pi) \cup (\pi, 3\pi)$. (Give exact values.)

e. Sketch the graph of $y = h(x)$ on the axes below.

- Give the exact coordinates of any stationary points.
- Label each asymptote with its equation.
- Give the exact value of the y-intercept.

Q7 (2009 VCAA Exam 1)

Let $f: R \to R$, $f(x) = e^x + k$, where k is a real number. The tangent to the graph of f at the point where $x = a$ passes through the point $(0, 0)$. Find the value of k in terms of a.

Q8 (2009 VCAA Exam 2)

At the point $(1, 1)$ on the graph of the function with rule $y = (x-1)^3 + 1$

A. there is a local maximum.

B. there is a local minimum.

C. there is a stationary point of inflection.

D. the gradient is not defined.

E. there is a point of discontinuity.

Q9 (2009 VCAA Exam 2)

The tangent at the point $(1, 5)$ on the graph of the curve $y = f(x)$ has equation $y = 3 + 2x$.

The tangent at the point $(3, 8)$ on the curve $y = f(x-2) + 3$ has equation

A. $y = 2x - 4$

B. $y = x + 5$

C. $y = -2x + 14$

D. $y = 2x + 4$

E. $y = 2x + 2$

Q10 (2009 VCAA Exam 2)

A cubic function has the rule $y = f(x)$. The graph of the derivative function f' crosses the x-axis at $(2, 0)$ and $(-3, 0)$. The maximum value of the derivative function is 10.

The value of x for which the graph of $y = f(x)$ has a local maximum is

A. -2

B. 2

C. -3

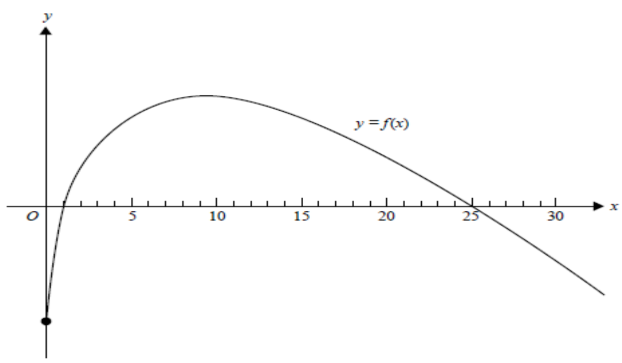
D. 3

E. $-\frac{1}{2}$

Q11 (2009 VCAA Exam 2)

Let $f: R^+ \cup \{0\} \to R$, $f(x) = 6\sqrt{x} - x - 5$.

The graph of $y = f(x)$ is shown below.



State the interval for which the graph of f is strictly decreasing.