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Free maths self-help revision

Composition of functions

More functions can be generated by addition/subtraction, multiplication/division and transformations of functions. Composition of functions creates new functions called composite functions.

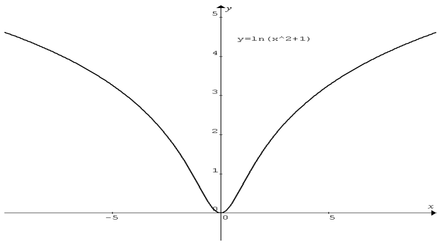
Given two functions f and g, f composition g, denoted as $f \circ g$ *, is defined by* $f(g(x))$ *when the range of g is a subset of the domain of f, i.e.* $r_g \subseteq d_f$.

Example 1 Given $f(x)=\log_e(x)$ and $g(x)=x^2+1$, find the rule for $f \circ g$ and state its domain and range.

$f \circ g(x)=f(g(x))=\log_e(g(x))=\log_e(x^2+1)$

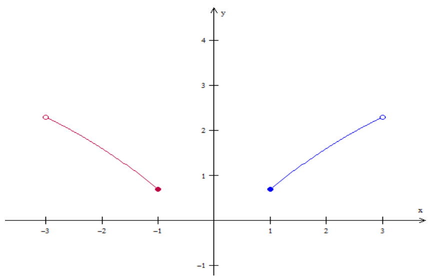
The maximal domain of *f* is $(0,\infty)$, .: $x^2+1>0$ and it is true for any $x \in R$ . .: the domain of $f \circ g$ is *R*.

Since $x^2+1 \ge 1$, .: the lowest value of $f \circ g$ is $\log_e 1=0$, .: the range of $f \circ g$ is $[0,\infty)$.



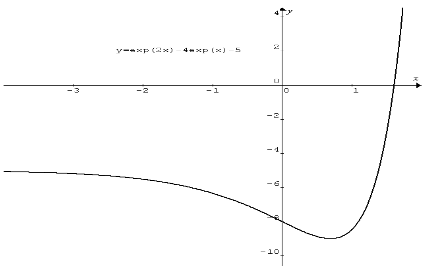
Example 2 Given $f:[2,10) \to R$, $f(x)=\log_e(x)$ and $g: D \to R$, $g(x)=x^2+1$, find the maximal domain *D* such that $f \circ g$ is defined and state the range of $f \circ g$.

The domain of *f* is $[2,10)$, .: $2 \le x^2+1<10$, $1 \le x^2<9$
.: $-3<x \le -1$ or $1 \le x<3$
.: *D* is $(-3,-1] \cup [1,3)$ which is also the domain of $f \circ g$.
The range of $f \circ g$ is $[\log_e 2, \log_e 10)$.



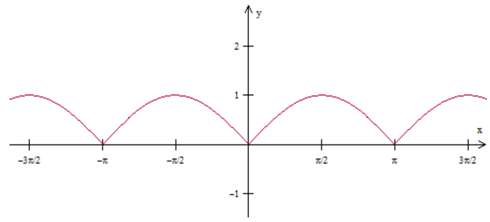
Example 3 Given $f(x)=e^x$ and $g(x)=x^2-4x-5$, find the rule for $g \circ f$ and state its domain.

$g \circ f(x)=g(f(x))=(e^x)^2-4e^x-5=e^{2x}-4e^x-5$
r_f is $(0,\infty)$, d_g is *R*, .: $r_f \subseteq d_g$.: the domain of *f* is also the domain of $g \circ f$, i.e. *R*.

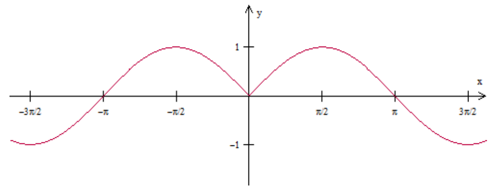


Example 4 Given $f(x)=|x|$ and $g(x)=\sin x$, find (a) $f \circ g$ and (b) $g \circ f$. Sketch the graph and state the range in each case.

(a) $f \circ g(x)=f(g(x))=f(\sin x)=|\sin x|$. The range is $[0,1]$.

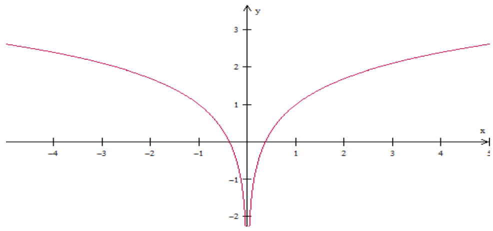


(b) $g \circ f(x)=g(f(x))=g(|x|)=\sin|x|$. The range is $[-1,1]$.



Example 5 (2006 VCAA Sample Exam 2 Version 2)
Find the maximal domain, *D*, of the function $f: D \to R$ with rule $f(x)=\log_e(|x|)+1$.

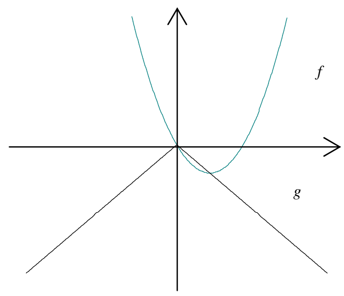
$|x|>0$, i.e. $x \in R \setminus \{0\}$, .: *D* is $R \setminus \{0\}$.



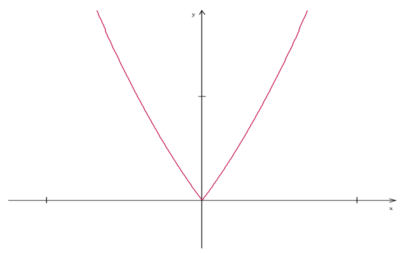
Example 6 (2007 VCAA Exam 2) Let $g(x)=x^2+2x-3$ and $f(x)=e^{2x+3}$. Find $f(g(x))$.

$f(g(x))=f(x^2+2x-3)=e^{2(x^2+2x-3)+3}=e^{2x^2+4x-3}$

Example 7 (2007 VCAA Exam 2) The graphs of $y=f(x)$ and $y=g(x)$ are shown below. Sketch the graph of $y=f(g(x))$.



Let $f(x)=x(x-2)$ and $g(x)=-|x|$.
$y=f(g(x))=-|x|(-|x|-2)=|x|(|x|+2)$
For $x \ge 0$, $y=x(x+2)$; for $x<0$, $y=-x(-x+2)=x(x-2)$



Example 8 (2010 VCAA Exam 2) Given $f(x)=\frac{1}{2}e^{3x}$ and $g(x)=\log_e(2x)+3$, find $g(f(x))$ and its maximal domain.

$g(f(x))=g\left(\frac{1}{2}e^{3x}\right)=\log_e\left(2\times\frac{1}{2}e^{3x}\right)+3$

$=\log_e e^{3x}+3=3x+3$.

The range of *f* is a subset of the domain of *g*, i.e. $r_f \subseteq d_g$,

.: the maximal domain of $g(f(x))$ is the maximal domain of *f*, i.e. *R*.

Example 9 Let $f:(-1,\infty)\to R$, $f(x)=p-x^2$ and $g:(-\infty,2]\to R$, $g(x)=\frac{1}{(x-3)^2}-p$, where $p>0$. Find the values of *p* such that both $f\circ g$ and $g\circ f$ are defined.

$d_f=(-1,\infty)$, $r_f=(-\infty,p]$, why?

$d_g=(-\infty,2]$, $r_g=(-p,1-p]$, why?

For $f\circ g$ to be defined, $r_g\subseteq d_f$, i.e. $(-p,1-p]\subseteq(-1,\infty)$

.: $-p\geq -1$, i.e. $p\leq 1$

For $g\circ f$ to be defined, $r_f\subseteq d_g$, i.e. $(-\infty,p]\subseteq(-\infty,2]$

.: $p\leq 2$

.: for both to be defined, $p\leq 1$ AND $p\leq 2$, .: $0<p\leq 1$

Functional equations

$2x^2-3y=5$ *is an equation in x and y.*

$3f(x)+(f(y))^2-1=0$ *is a functional equation involving* $f(x)$ *and* $f(y)$.

Example 10 Show that $f(x)=x+\frac{1}{x}$ satisfies the functional equation $f(uv)+f\left(\frac{u}{v}\right)=f(u)f(v)$ for $u,v\neq 0$.

$f(x)=x+\frac{1}{x}$, $f(uv)=uv+\frac{1}{uv}$, $f\left(\frac{u}{v}\right)=\frac{u}{v}+\frac{v}{u}$

.: $f(uv)+f\left(\frac{u}{v}\right)=uv+\frac{1}{uv}+\frac{u}{v}+\frac{v}{u}$

$f(u)f(v)=\left(u+\frac{1}{u}\right)\left(v+\frac{1}{v}\right)=uv+\frac{1}{uv}+\frac{u}{v}+\frac{v}{u}$

.: $f(uv)+f\left(\frac{u}{v}\right)=f(u)f(v)$.

Example 11 Given $f(x)=\log_e x+1$, show that $f\left(\frac{x}{y}\right)-1=f(x)-f(y)$.

$f(x)=\log_e x+1$, $f(y)=\log_e y+1$

.: $f(x)-f(y)=\log_e x-\log_e y=\log_e\frac{x}{y}$

$f\left(\frac{x}{y}\right)-1=\log_e\frac{x}{y}+1-1=\log_e\frac{x}{y}$.: $f\left(\frac{x}{y}\right)-1=f(x)-f(y)$.

Example 12 (2008 VCAA Exam 2)
Let $f:R\to R$, $f(x)=e^x+e^{-x}$. For all $u\in R$, express $f(2u)$ in terms of $f(u)$.

$f(x)=e^x+e^{-x}$, .: $f(2u)=e^{2u}+e^{-2u}=e^{2u}+2+e^{-2u}-2$

$=(e^u+e^{-u})^2-2=(f(u))^2-2$

Example 13 (2009 VCAA Exam 2) Let $f:R\to R$, $f(x)=x^2$. Which one of the following is **not** true? A. $f(xy)=f(x)f(y)$
B. $f(x)-f(-x)=0$ C. $f(x-y)=f(x)-f(y)$
D. $f(2x)=4f(x)$ E. $f(x+y)+f(x-y)=2(f(x)+f(y))$

$f(x-y)=(x-y)^2=x^2+y^2-2xy$, $f(x)-f(y)=x^2-y^2$

.: $f(x-y)\neq f(x)-f(y)$

.: C is not true.

Example 14 (2006 VCAA Exam 2) The function *f* satisfies the functional equation $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ where *x* and *y* are any non-zero real numbers. Which one of the following rules is a possibility for the function? A. $f(x)=\log_e|x|$ B. $f(x)=\frac{1}{x}$
C. $f(x)=2^x$ D. $f(x)=2x$ E. $f(x)=\sin(2x)$

$f(x)=2x$ is a possible rule.

Check: $f\left(\frac{x+y}{2}\right)=2\left(\frac{x+y}{2}\right)=x+y$

$\frac{f(x)+f(y)}{2}=\frac{2x+2y}{2}=x+y$

.: $f(x)=2x$ satisfies $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$.

Example 15 (2007 VCAA Exam 2) The function *f* satisfies the functional equation $f(f(x))=x$ for the maximal domain of *f*. Which of the following rules is a possibility for *f*?
A. $f(x)=x+1$ B. $f(x)=x-1$ C. $f(x)=\frac{x-1}{x+1}$
D. $f(x)=\log_e x$ E. $f(x)=\frac{x+1}{x-1}$

$f(x)=\frac{x+1}{x-1}$, $f(f(x))=\frac{f(x)+1}{f(x)-1}=\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}=\frac{2x}{2}=x$

.: $f(x)=\frac{x-1}{x+1}$ satisfies the functional equation.

Comment: $f(f(x))=x$, .: $f(x)=f^{-1}(x)$, i.e. the function and its inverse are the same. For $f(x)=\frac{x+1}{x-1}$, $f^{-1}(x)=\frac{x+1}{x-1}$, it satisfies the requirement $f(x)=f^{-1}(x)$.

Example 16 For $f(x)=1-\sqrt{x}$, show that
(a) $f(xy)=f(x)+f(y)-f(x)f(y)$ and
(b) $f\left(\frac{x}{y}\right)=\frac{f(x)-f(y)}{1-f(y)}$ for $x,y\in R^+$.

(a) $f(x)=1-\sqrt{x}$, $f(y)=1-\sqrt{y}$

$f(x)+f(y)-f(x)f(y)=1-\sqrt{x}+1-\sqrt{y}-(1-\sqrt{x})(1-\sqrt{y})$

$=2-\sqrt{x}-\sqrt{y}-(1-\sqrt{x}-\sqrt{y}+\sqrt{x}\sqrt{y})=1-\sqrt{xy}=f(xy)$

(b) $\frac{f(x)-f(y)}{1-f(y)}=\frac{(1-\sqrt{x})-(1-\sqrt{y})}{1-(1-\sqrt{y})}=\frac{\sqrt{y}-\sqrt{x}}{\sqrt{y}}$

$=1-\frac{\sqrt{x}}{\sqrt{y}}=1-\sqrt{\frac{x}{y}}=f\left(\frac{x}{y}\right)$

Exercise: Next page

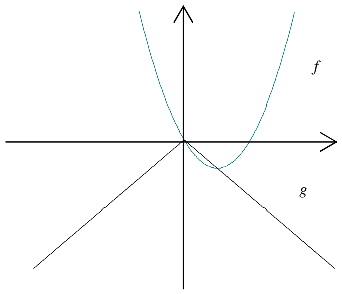
Q1 Given $f(x)=e^x$ and $g(x)=x^2+1$, find the rule for $f \circ g$ and state its domain and range.

Q2 Given $f:[1,2)\to R$, $f(x)=e^x$ and $g:D\to R$, $g(x)=x^2+1$, find the maximal domain D such that $f \circ g$ is defined and state the range of $f \circ g$.

Q3 Given $f(x)=|x-1|$ and $g(x)=\sin x$, find (a) $f \circ g$ and (b) $g \circ f$. Sketch the graph and state the range in each case.

Q4 Let $g(x)=x^2+2x-3$ and $f(x)=e^{2x+3}$. Find $g(f(x))$ and its domain and range.

Q5 The graphs of $y=f(x)$ and $y=g(x)$ are shown below. Sketch the graph of $y=g(f(x))$.



Q6 Let $f:(1,\infty)\to R$, $f(x)=p-x^2$ and $g:(-\infty,2)\to R$, $g(x)=\dfrac{1}{(x-2)^2}+p$, where $p>0$. Find the values of p such that both $f \circ g$ and $g \circ f$ are defined.

Q7 Show that $f(x)=x-\dfrac{1}{x}$ satisfies the functional equation $f\left(\dfrac{v}{u}\right)+f\left(\dfrac{u}{v}\right)=0$ for $u,v\neq 0$.

Q8 Given $f(x)=1-\log_e x$, show that $f(x)+f(y)=1-f(xy)$.

Q9 Given $f(x)=e^x-e^{-x}$ and $f'(x)=e^x+e^{-x}$ express $f(2x)$ in terms of $f(x)$ and $f'(x)$.

Q10 Show that $f(x)=\dfrac{x}{x-1}$ satisfies the functional equation $f(f(x))=x$.

Numerical, algebraic and worded answers: 1. Rule: $f(g(x))=e^{x^2+1}$, domain: *R*, range: $[e,\infty)$ 2. *D* is $(-1,1)$, range: $[e,e^2)$ 3a. $[0,2]$ 3b. $[-1,1]$
4. $g(f(x))=e^{4x+6}+2e^{2x+3}-3$, *R*, $(-3,\infty)$ 6. $1<p<3$ 8. $f(x)+f(y)=1-\log_e x+1-\log_e y=1+1-\log_e xy=1+f(xy)$
7. $f\left(\dfrac{u}{v}\right)+f\left(\dfrac{v}{u}\right)=\dfrac{u}{v}-\dfrac{v}{u}+\dfrac{v}{u}-\dfrac{u}{v}=0$ 9. $f(2x)=e^{2x}-e^{-2x}=(e^x-e^{-x})(e^x+e^{-x})=f(x)f'(x)$ 10. $f(f(x))=\dfrac{f(x)}{f(x)-1}=\dfrac{\frac{x}{x-1}}{\frac{x}{x-1}-1}=\dfrac{x}{1}=x$