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Algebraic solution of equations involving circular functions

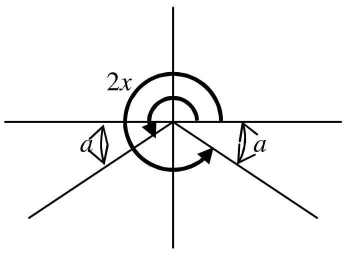
It is wise to consider the domain and change it to the restriction on the argument when solving an equation involving one or more circular functions.
In solving such equations algebraically, always transpose an equation to make $\sin(kx)$, $\cos(kx)$ *or* $\tan(kx)$ *the subject of the equation.*

Example 1 Solve $3\sin(2x)=-2$, where $0<x<6$.

The domain is $0<x<6$, or $0<2x<12$.

$3\sin(2x)=-2$, $\sin(2x)=-\frac{2}{3}$.

Since $\sin(2x)$ has a negative value, $-\frac{2}{3}$, then the 'angle' $2x$ must be in the third or fourth quadrant.



$a=\sin^{-1}\left(\frac{2}{3}\right)=0.7297$ (Calculator)

$2x=\pi+0.7297$, $2\pi-0.7297$, $3\pi+0.7297$, $4\pi-0.7297$

$2x=3.8713, 5.5535, 10.1545, 11.8366$

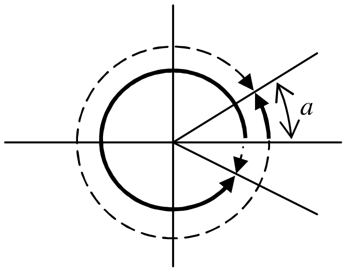
$\therefore x=1.9357, 2.7767, 5.0772, 5.9183$

Example 2 Find the exact solution(s) of $2\sqrt{3}\cos\left(\frac{\theta}{2}\right)-3=0$, where $-4\pi<\theta<4\pi$.

The domain is $-4\pi<\theta<4\pi$, or $-2\pi<\frac{\theta}{2}<2\pi$.

$2\sqrt{3}\cos\left(\frac{\theta}{2}\right)-3=0$, $\cos\left(\frac{\theta}{2}\right)=\frac{\sqrt{3}}{2}$.

Since $\cos\left(\frac{\theta}{2}\right)$ has a positive value, $\frac{\sqrt{3}}{2}$, $\therefore$ the 'angle' $\frac{\theta}{2}$ must be in the first or fourth quadrant.



$a=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$

$\frac{\theta}{2}=-2\pi+\frac{\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, 2\pi-\frac{\pi}{6}$

$\therefore \frac{\theta}{2}=-\frac{11\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}$ $\quad \therefore \theta=-\frac{11\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{11\pi}{3}$.

Example 3 Find the exact coordinates of the intersections of $y=3\sin(2x)+1$ and $y=\sqrt{3}\cos(2x)+1$, where $-\pi<x<\pi$.

Solve the two equations simultaneously to find the intersections.

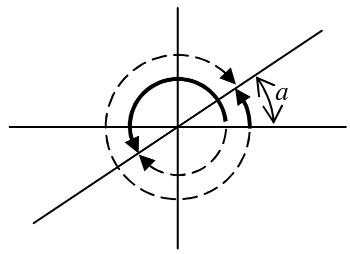
$y=3\sin(2x)+1$ ………(1) $\qquad y=\sqrt{3}\cos(2x)+1$ ……(2)

The domain is $-\pi<x<\pi$, or $-2\pi<2x<2\pi$.

Substitute eq(1) in eq(2), $3\sin(2x)+1=\sqrt{3}\cos(2x)+1$,

$\therefore 3\sin(2x)=\sqrt{3}\cos(2x)$, $\frac{\sin(2x)}{\cos(2x)}=\frac{\sqrt{3}}{3}$, $\tan(2x)=\frac{1}{\sqrt{3}}$.

Since $\tan(2x)$ has a positive value, $\frac{1}{\sqrt{3}}$, the 'angle' $2x$ must be in the first or third quadrant.



$a=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$

$2x=-2\pi+\frac{\pi}{6}, -\pi+\frac{\pi}{6}, \frac{\pi}{6}, \pi+\frac{\pi}{6}$,

$\therefore 2x=-\frac{11\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{6}, \frac{7\pi}{6}$, $\therefore x=-\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$

Substitute each of these (2x) values in eq(1),

$y=3\sin\left(-\frac{11\pi}{6}\right)+1=3\left(\frac{1}{2}\right)+1=\frac{5}{2}$,

$y=3\sin\left(-\frac{5\pi}{6}\right)+1=3\left(-\frac{1}{2}\right)+1=-\frac{1}{2}$,

$y=3\sin\left(\frac{\pi}{6}\right)+1=3\left(\frac{1}{2}\right)+1=\frac{5}{2}$,

$y=3\sin\left(\frac{7\pi}{6}\right)+1=3\left(-\frac{1}{2}\right)+1=-\frac{1}{2}$.

The coordinates of the intersections are:

$\left(-\frac{11\pi}{12},\frac{5}{2}\right)$, $\left(-\frac{5\pi}{12},-\frac{1}{2}\right)$, $\left(\frac{\pi}{12},\frac{5}{2}\right)$ and $\left(\frac{7\pi}{12},-\frac{1}{2}\right)$.

Example 4 Solve $\sin^2(x)=\sin(x)\cos(x)$ for x, where $0\le x\le\frac{\pi}{2}$. Note: $\sin^2(x)$ is the proper notation for $(\sin(x))^2$.

Note: Do not divide both sides of the equation by $\sin(x)$, because $\sin(x)=0$ is a possibility. See below.

$\sin^2(x)=\sin(x)\cos(x)$, $\sin^2(x)-\sin(x)\cos(x)=0$, common factor, $\sin(x)(\sin(x)-\cos(x))=0$.

$\therefore$ either $\sin(x)=0$, $\therefore x=0$

OR $\sin(x)-\cos(x)=0$, $\frac{\sin(x)-\cos(x)}{\cos(x)}=\frac{0}{\cos(x)}$ for $\cos(x)\ne0$,

$\therefore \tan(x)-1=0$, $\tan(x)=1$, $x=\frac{\pi}{4}$.

The two possible solutions for x are 0, $\frac{\pi}{4}$.

Example 5 (2007 VCAA Exam 1)

Solve the equation $\sin\left(\frac{2\pi x}{3}\right)=-\frac{\sqrt{3}}{2}$ for $x\in[0,3]$.

The domain is $0\le x\le 3$, .: $0\le\frac{2\pi x}{3}\le 2\pi$

.: $\frac{2\pi x}{3}=\frac{4\pi}{3}$ or $\frac{5\pi}{3}$, .: $x=2$ or $\frac{5}{2}$

Example 6 (2007 VCAA Exam 2)
Solve $\cos^2 x+2\cos x=0$ for $x\in(-2\pi,2\pi)$.

$\cos^2 x+2\cos x=0$, $\cos x(\cos x+2)=0$

Since $\cos x+2\ne 0$, .: $\cos x=0$, $x=-\frac{3\pi}{2}$, $-\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$

Example 7 Find t when $x=10\sin\left(\frac{\pi t}{12}\right)-12$ is a minimum, where $0<t<48$.

The domain is $0<t<48$, or $0<\frac{\pi t}{12}<4\pi$.

Minimum of x is -22, it occurs when $\sin\left(\frac{\pi t}{12}\right)=-1$,

$\therefore\frac{\pi t}{12}=\frac{3\pi}{2}$, $\frac{7\pi}{2}$. $\therefore t=18$, 42.

Example 8 (2006 VCAA Sample Exam 2 Version 2)
The height , h metres, of point P on a Ferris wheel above ground level, at time t hours after 1:00 pm is given by

$$h(t)=62+60\sin\left(\frac{(5t-1)\pi}{2}\right).$$

Point P was at its lowest point at 1:00 pm.
(a) At what time, after 1:00 pm, does point P first return to its lowest point?
(b) Find the time, after 1:00 pm, when point P first reaches a height of 92 metres above ground level.
(c) Find the number of minutes during one rotation when point P is at least 92 metres above ground level.

(a) *The period of rotation* $=\frac{2\pi}{n}=\frac{2\pi}{\frac{5\pi}{2}}=0.8$ hour, i.e. 48 minutes

.: 1:48 pm

(b) $62+60\sin\left(\frac{(5t-1)\pi}{2}\right)=92$, $60\sin\left(\frac{(5t-1)\pi}{2}\right)=30$

$\sin\left(\frac{(5t-1)\pi}{2}\right)=\frac{1}{2}$, $\frac{(5t-1)\pi}{2}=\frac{\pi}{6}$, $5t-1=\frac{1}{3}$

.: $t=\frac{4}{15}$ hour, i.e. 16 minutes, .: 1:16 pm

(c) Second time at 92 metres: $\frac{(5t-1)\pi}{2}=\frac{5\pi}{6}$, $5t-1=\frac{5}{3}$

.: $t=\frac{8}{15}$ hour, i.e. 32 minutes, .: 1:32 pm

.: *time interval* $=16$ minutes

Example 9 (2013 VCAA Exam 2)
During a particular 24-hour time interval, the temperature (T° C) is given by $T(t)=25+2\cos\left(\frac{\pi t}{8}\right)$, $0\le t\le 24$. For how many hours during the 24-hour time interval is $T\ge 26$?

Let $25+2\cos\left(\frac{\pi t}{8}\right)=26$. .: $\cos\left(\frac{\pi t}{8}\right)=\frac{1}{2}$ where $0\le\frac{\pi t}{8}\le 3\pi$

.: $\frac{\pi t}{8}=\frac{\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{3}$, .: $t=\frac{8}{3}$, $\frac{40}{3}$, $\frac{56}{3}$

Number of hours that $T\ge 26=\left(\frac{56}{3}-\frac{40}{3}\right)+\left(\frac{8}{3}-0\right)=8$

General solutions

If there is no restriction placed on the domain, the maximal domain is assumed and the solution is given in general form.
For $\tan x=a$, $x=n\pi+\tan^{-1}a$
For $\cos x=a$, $x=2n\pi\pm\cos^{-1}a$
For $\sin x=a$, $x=2n\pi+\sin^{-1}a$ or $2n\pi+\pi-\sin^{-1}a$
where n is an integer.

Example 10 (2009 VCAA Exam 2) Find the general solution to the equation $\sin(2x)=-1$.

$\sin(2x)=-1$, $2x=2n\pi+\sin^{-1}(-1)=2n\pi-\frac{\pi}{2}$

.: $x=n\pi-\frac{\pi}{4}$ where n is an integer

Equivalent forms (compound and double angle formulas)

Compound angle: $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

Double angle: $\sin 2A=2\sin A\cos A$, $\cos 2A=\cos^2 A-\sin^2 A$
or $\cos 2A=2\cos^2 A-1$ ***or*** $\cos 2A=1-2\sin^2 A$

$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$

Example 11 Show (not by substitution) that a solution to $\tan^2 x+2\tan x-1=0$ is $x=\frac{\pi}{8}$.

$\tan^2 x+2\tan x-1=0$, $2\tan x=1-\tan^2 x$, $\frac{2\tan x}{1-\tan^2 x}=1$

.: $\tan 2x=1$, $2x=\frac{\pi}{4}$, $x=\frac{\pi}{8}$.

Example 12 Solve $\cos 2x+\cos x=0$ for $x\in[0,2\pi]$.

$\cos 2x+\cos x=0$, $2\cos^2 x+\cos x-1=0$,
$(2\cos x-1)(\cos x+1)=0$, .: $2\cos x-1=0$ or $\cos x+1=0$

.: $\cos x=\frac{1}{2}$ or $\cos x=-1$.: $x=\frac{\pi}{3}$, $\frac{5\pi}{3}$ or π

Example 13 Solve $\cos\frac{\pi}{12}\cos x+\sin\frac{\pi}{12}\sin x=\frac{1}{\sqrt{2}}$, $x\in[-\pi,\pi]$

$\cos\frac{\pi}{12}\cos x+\sin\frac{\pi}{12}\sin x=\frac{1}{\sqrt{2}}$, $\cos\left(\frac{\pi}{12}-x\right)=\frac{1}{\sqrt{2}}$

.: $\frac{\pi}{12}-x=\pm\frac{\pi}{4}$, $x=\frac{\pi}{12}\mp\frac{\pi}{4}$, $x=-\frac{\pi}{6}$, $\frac{\pi}{3}$

Example 14 Given $\cos a=\frac{3}{5}$ where $0<a<\frac{\pi}{2}$, find the exact value of $\sin\left(a+\frac{\pi}{3}\right)$.

$\cos a=\frac{3}{5}$, .: $\sin a=\sqrt{1-\cos^2 a}=\sqrt{1-\frac{9}{25}}=\frac{4}{5}$

$\sin\left(a+\frac{\pi}{3}\right)=\sin a\cos\frac{\pi}{3}+\cos a\sin\frac{\pi}{3}$

$=\frac{4}{5}\times\frac{1}{2}+\frac{3}{5}\times\frac{\sqrt{3}}{2}=\frac{4+3\sqrt{3}}{10}$

Exercise: Next page

Q1 Find the exact solution(s) of $2\cos\left(\frac{x}{2}\right)+\sqrt{2}=0$, where $-2\pi < x < 2\pi$.

Q2 Solve $5\sin(2x)=1$, where $0 < x < 6$.

Q3 Find the exact coordinates of the intersections of $y=\sqrt{3}\sin(x)-1$ and $y=3\cos(x)-1$, where $-\pi < x < \pi$.

Q4 Solve $\cos^2(x)=\sin(x)\cos(x)$ for x, where $0 \le x \le \frac{\pi}{2}$.

Q5 Solve the equation $\cos\left(\frac{2\pi x}{3}\right)=\frac{\sqrt{3}}{2}$ for $x \in [0,3]$.

Q6 Solve $\sin^2(\pi x)+2\sin(\pi x)=0$ for $x \in (-2,2)$.

Q7 Find t when $x=10\sin\left(\frac{\pi t}{6}\right)+12$ is a maximum, where $0 < t < 48$.

Q8 Find the general solution to the equation $\cos(2x)=-1$.

Q9 Find the general solution to the equation $\sin x=\frac{1}{2}$.

Q10 Solve $\sin 2x+\sin x=0$ for $x \in [0,2\pi]$.

Numerical, algebraic and worded answers: 1. $x=-3\pi/4$, $3\pi/4$ 2. $x \approx 0.1007$, 1.4701, 3.2423, 4.6117 3. $x=-2\pi/3$, $\pi/3$
4. $x=\pi/4$, $\pi/2$ 5. $x=1/4$, $11/4$ 6. $x=-2$, -1, 0, 1, 2 7. $t=3$, 15, 27, 39 8. $x=(2n+1)\pi/2$, can also be $(2n-1)\pi/2$
9. $x=2n\pi+\pi/6$, $(2n+1)\pi-\pi/6$ 10. $x=0$, $2\pi/3$, π, $4\pi/3$, 2π