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Exponential (index) laws

1. $a^m \times a^n = a^{m+n}$

2. $\frac{1}{a^n} = a^{-n}$; $a^m = \frac{1}{a^{-m}}$; $\frac{a^m}{a^n} = a^{m-n}$

3. $(a^m)^n = a^{mn}$ 4. $(ab)^n = a^n b^n$

Handy knowledge: $a^0 = 1$, $a^{\frac{1}{p}} = \sqrt[p]{a}$, $a^{\frac{q}{p}} = \left(\sqrt[p]{a}\right)^q$ or $\sqrt[p]{a^q}$.

Example 1 Simplify $\frac{(ab)^{2n} - b^{3n}}{(ab^4)^n + b^{4n}}$, express in positive indices.

$$\frac{(ab)^{2n} - b^{4n}}{(ab^4)^n + b^{5n}} = \frac{a^{2n}b^{2n} - b^{4n}}{a^n b^{4n} + b^{5n}} = \frac{b^{2n}\left(a^{2n} - b^{2n}\right)}{b^{4n}\left(a^n + b^n\right)}$$

$$= \frac{b^{-2n}\left((a^n)^2 - (b^n)^2\right)}{(a^n + b^n)} = \frac{b^{-2n}(a^n - b^n)(a^n + b^n)}{(a^n + b^n)} = \frac{a^n - b^n}{b^{2n}} .$$

Example 2 Simplify $\frac{e^{2x+1} - 4e^{x+1} + 3e}{e^{x+1} - e}$.

$$\frac{e^{2x+1} - 4e^{x+1} + 3e}{e^{x+1} - e} = \frac{e\left(e^{2x} - 4e^x + 3\right)}{e\left(e^x - 1\right)} = \frac{e^0\left((e^x)^2 - 4(e^x) + 3\right)}{(e^x - 1)}$$

$$= \frac{(e^x - 3)(e^x - 1)}{(e^x - 1)} = e^x - 3 \text{ for } x \ne 0 .$$

Note: It is necessary to state that $x \ne 0$ because $\frac{e^{2x+1} - 4e^{x+1} + 3e}{e^{x+1} - e}$ is undefined for $x = 0$ whilst $e^x - 3$ is defined for all *x*.

Example 3 Simplify $\frac{p^{\frac{5}{2}} + 2p^{\frac{3}{2}}}{5p^{\frac{3}{2}} + 10p^{\frac{1}{2}}}$.

$$\frac{p^{\frac{5}{2}} + 2p^{\frac{3}{2}}}{5p^{\frac{3}{2}} + 10p^{\frac{1}{2}}} = \frac{p^{\frac{3}{2}}(p+2)}{5p^{\frac{1}{2}}(p+2)} = \frac{p^{\frac{3}{2}-\frac{1}{2}}}{5} = \frac{p}{5} \text{ for } p \ne -2 .$$

Example 4 Simplify $f(n) = \left(100^n + 2\times 10^{n+1} + 100\right)^{\frac{3}{2}}$. Show that $f(2) = 1000 f(0)$.

$$\left(100^n + 2\times 10^{n+1} + 100\right)^{\frac{3}{2}} = \left((10^2)^n + 2\times 10 \times 10^n + 10^2\right)^{\frac{3}{2}}$$

$$= \left((10^n)^2 + 2(10)(10^n) + 10^2\right)^{\frac{3}{2}} = \left((10^n + 10)^2\right)^{\frac{3}{2}} = (10^n + 10)^3$$

$\therefore$ $f(n) = (10^n + 10)^3$. $f(0) = (10^0 + 10)^3 = 11^3$

$$f(2) = (10^2 + 10)^3 = (110)^3 = (10\times 11)^3 = 10^3 \times 11^3 = 1000 f(0) .$$

Example 5 Simplify $\frac{6\times\sqrt[3]{2x^5y^2}}{\sqrt{(6x^6y)^{\frac{2}{3}}}}$, express in positive indices.

$$\frac{6\times\sqrt[3]{2x^5y^2}}{\sqrt{(6x^6y)^{\frac{2}{3}}}} = \frac{6\left(2x^5y^2\right)^{\frac{1}{3}}}{\left((6x^6y)^{\frac{2}{3}}\right)^{\frac{1}{2}}} = \frac{2\times 3\times 2^{\frac{1}{3}} x^{\frac{5}{3}} y^{\frac{2}{3}}}{(2\times 3x^6 y)^{\frac{1}{3}}}$$

$$= \frac{3\times 2^{\frac{4}{3}} x^{\frac{5}{3}} y^{\frac{2}{3}}}{2^{\frac{1}{3}}\times 3^{\frac{1}{3}} x^2 y^{\frac{1}{3}}} = 2\times 3^{\frac{2}{3}} x^{-\frac{1}{3}} y^{\frac{1}{3}} = \frac{2\times 3^{\frac{2}{3}} y^{\frac{1}{3}}}{x^{\frac{1}{3}}} .$$

Logarithm laws

For $p, q > 0$,

1. $\log_a p + \log_a q = \log_a(pq)$

2. $\log_a p - \log_a q = \log_a\left(\frac{p}{q}\right)$; $-\log_a q = \log_a\left(\frac{1}{q}\right)$

3. $\log_a p^n = n\log_a p$

4. $\log_b p = \frac{\log_a p}{\log_a b}$

Law 4 shows the relationship between $\log_a p$ *and* $\log_b p$.

Handy knowledge: $\log_a 1 = 0$ *;* $\log_a a = 1$ *;*

$\log_a p$ *is undefined for* $p \le 0$ *;*

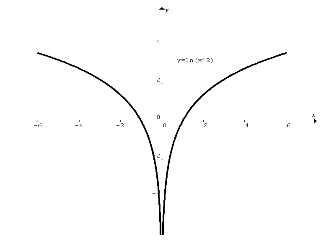
$\log_a p < 0$ *for* $0 < p < 1$ *;* $\log_a p > 0$ *for* $p > 1$.

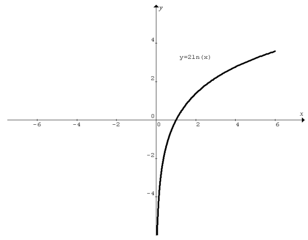
For even n, $\log_a p^n$ *is defined for all* $p \in R$ *whilst* $n\log_a p$ *is defined only for* $p > 0$,

$\therefore \log_a p^n = n\log_a p$ *for* $p > 0$, *and*

$\log_a p^n \ne n\log_a p$ *for* $p \le 0$.

The following graphs of $y = \log_e x^2$ *and* $y = 2\log_e x$ *illustrate the point.*





Example 6 Evaluate $5\log_2\left(\frac{1}{32}\right)$.

$$5\log_2\left(\frac{1}{32}\right) = 5\log_2\left(\frac{1}{2^5}\right) = 5\log_2 2^{-5} = -5\times 5\log_2 2 = -25 .$$

Example 7 Simplify $2\log_{10}(3x^2y) - 3\log_{10}(2xy^2)$.

$$2\log_{10}(3x^2y) - 3\log_{10}(2xy^2) = \log_{10}(3x^2y)^2 - \log_{10}(2xy^2)^3$$

$$= \log_{10}(9x^4y^2) - \log_{10}(8x^3y^6)$$

$$= \log_{10}\left(\frac{9x^4y^2}{8x^3y^6}\right) = \log_{10}\left(\frac{9x}{8y^4}\right) \text{ for } y > 0 .$$

Example 8 Evaluate $\log_3 10$.

Graphics calculators have only log (i.e. $\log_{10}$) and ln (i.e. $\log_e$).

$\log_3 10 = \frac{\log_{10} 10}{\log_{10} 3} = 2.0959$, or $\log_3 10 = \frac{\log_e 10}{\log_e 3} = 2.0959$.

You can use CAS to evaluate $\log_3 10$ directly.

Example 9 Show that $3\log_4 x - 2\log_8 x = \frac{5}{6}\log_2 x$.

Change both logarithms on the left side of the identity to base 2.

$$3\log_4 x - 2\log_8 x = \frac{3\log_2 x}{\log_2 4} - \frac{2\log_2 x}{\log_2 8}$$

$$= \frac{3\log_2 x}{\log_2 2^2} - \frac{2\log_2 x}{\log_2 2^3} = \frac{3}{2}\log_2 x - \frac{2}{3}\log_2 x$$

$$= \left(\frac{3}{2} - \frac{2}{3}\right)\log_2 x = \frac{5}{6}\log_2 x$$

Example 10 Show that $\dfrac{1}{\log_e 10} = \log_{10} e$.

Change both sides to a common base b.

$$LHS = \frac{1}{\dfrac{\log_b 10}{\log_b e}} = \frac{\log_b e}{\log_b 10}, \quad RHS = \frac{\log_b e}{\log_b 10}. \quad \therefore LHS = RHS$$

Example 11 Given $10^p = e^q$, find (a) q in terms of p, (b) p in terms of q.

(a) $10^p = e^q$, $q = \log_e 10^p$, $\therefore$ $q = p\log_e 10$.

(b) Since $q = p\log_e 10$, $\therefore$ $p = \dfrac{q}{\log_e 10} = q\log_{10} e$.

These two results show the way to change the base of an exponential function.

$10^p = e^{p\log_e 10}$; $e^q = 10^{q\log_{10} e}$. In general, $a^x = b^{x\log_b a}$.

Example 12 Change 5^x to base 10 and base e.

$5^x = 10^{x\log_{10} 5} \approx 10^{0.6990x}$; $5^x = e^{x\log_e 5} \approx e^{1.6094x}$.

Equivalent relations

Examples are: $y = x^2 \Leftrightarrow x = \pm\sqrt{y}$;

$y = \sin(x) \Leftrightarrow x = \sin^{-1}(y)$;

$y = 10^x \Leftrightarrow x = \log_{10} y$;

$y = e^x \Leftrightarrow x = \log_e y$.

In each case, both left and right statements give exactly the same relationship between x and y, i.e. they are equivalent. Try to plot the graphs of a pair of equivalent relations. They are the same plot. The left relation uses y as the subject, and the right relation uses x. In the last two examples, the left relations are expressed in index (exponential) form whilst the right relations are in logarithm form.

Example 13 (2006 VCAA Exam 2) If $y = 3a^{2x} + b$, write x as the subject of the equation.

$$y = 3a^{2x} + b,\ 3a^{2x} = y - b,\ a^{2x} = \frac{y-b}{3},$$

$$2x = \log_a\left(\frac{y-b}{3}\right),\ x = \frac{1}{2}\log_a\left(\frac{y-b}{3}\right)$$

Example 14 (2007 VCAA Exam 2) If $y = \log_a(7x - b) + 3$, write x as the subject of the equation.

$$y = \log_a(7x-b) + 3,\ \log_a(7x-b) = y - 3,\ 7x - b = a^{y-3},$$

$$7x = a^{y-3} + b,\ x = \frac{1}{7}\left(a^{y-3} + b\right)$$

Inverse relations

$y = e^x$ *and* $x = \log_e y$ *are equivalent relations, but*

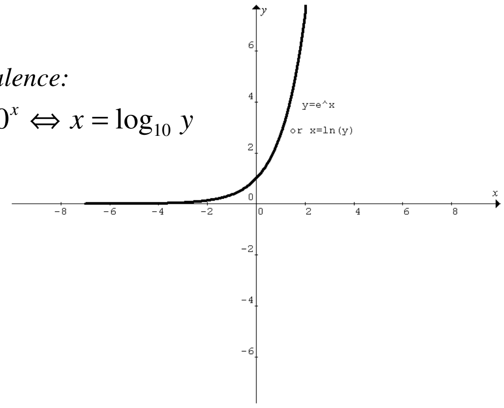
$y = e^x$ *and* $x = e^y$ *are inverse relations, and so are*

$y = \log_e x$ *and* $x = \log_e y$. *(Read each relation carefully)*

In inverse relations, the x and y-coordinates of all the points are interchanged.

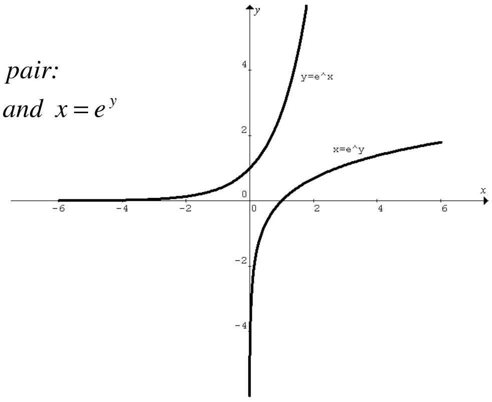
Equivalence:

$y = 10^x \Leftrightarrow x = \log_{10} y$



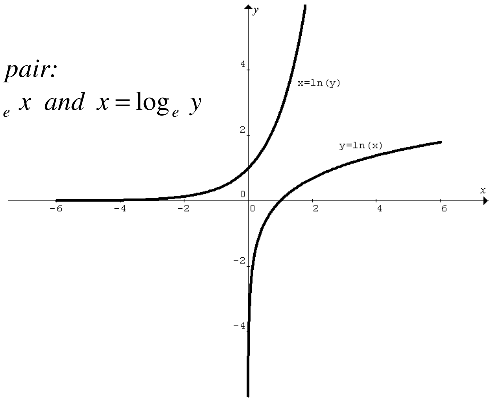
Inverse pair:

$y = e^x$ *and* $x = e^y$



Inverse pair:

$y = \log_e x$ *and* $x = \log_e y$

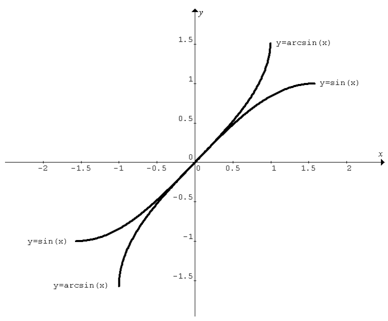


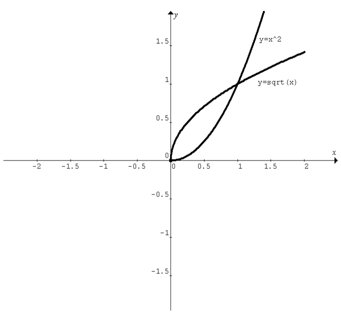
Other examples of inverse pairs are:

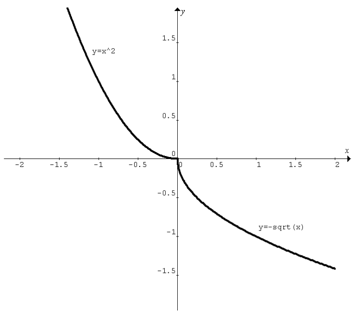
$y = \sin(x)$ *for* $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ *and* $y = \sin^{-1}(x)$;

$y = x^2$ *for* $x \ge 0$ *and* $y = \sqrt{x}$;

$y = x^2$ *for* $x < 0$ *and* $y = -\sqrt{x}$.







Exercise: Next page

Q1 Simplify $\frac{b^{2n}-(ab)^{3n}}{b^{4n}+(ab)^{4n}}$. Express your answer in positive indices.

Q2 Simplify $\frac{e^{1+x}+e^{1-x}-2e}{e^{x+1}-e}$. Express your answer in positive indices.

Q3 Simplify $\left(49^n+2\times 7^{n+1}+49\right)^{\frac{3}{2}}$.

Q4 Simplify $\frac{2\times\sqrt[3]{2x^2y^3}}{\sqrt{\left(2x^3y^2\right)^{\frac{2}{3}}}}$. Express your answer in positive indices.

Q5 Evaluate $\frac{1}{2}\log_5\left(\frac{1}{25}\right)$.

Q6 Simplify $\log_{10}\left(3x^2y\right)-2\log_{10}\left(6xy^2\right)$.

Q7 Change $\log_2 1000e$ to natural log.

Q8 If $y=2a^{3x}-b$, write x as the subject of the equation.

Q9 If $y=k\log_a(bx+c)-d$, write x as the subject of the equation.

Q10 Express $4\log_4 x-8\log_8 x$ in terms of $\log_2 x$.

Numerical, algebraic and worded answers: 1. $\frac{1-a^{3n}b^n}{b^{2n}\left(1+a^{4n}\right)}$ 2. $1-e^{-x}$ 3. $\left(7^n+7\right)^3$ 4. $2\left(\frac{y}{x}\right)^{\frac{1}{3}}$ 5. -1 6. $-\log_{10}\left(12y^3\right)$ 7. $\frac{1+\log_e 1000}{\log_e 2}$ 8. $x=\frac{1}{3}\log_a\left(\frac{y+b}{2}\right)$, 9. $x=\frac{1}{b}\left(a^{\frac{y+d}{k}}-c\right)$ 10. $-\frac{2}{3}\log_2 x$