VCAA Mathematical Methods 3/4 Exam 2 Solutions 2005

Q1ai $f(t)=2e^{-t}$, $t\geq 0$ is a decreasing function. At $t=0$, $y=f(0)=2$. As $t\to\infty$, $y\to 0^{+}$. $\therefore$ the range of *f* is $(0,2]$.

Q1aii Equation of *f*: $y=2e^{-t}$, equation of f^{-1}: $t=2e^{-y}$.

Re-express with *y* as the subject: $e^{-y}=\frac{t}{2}$, $e^{y}=\frac{2}{t}$,

$\therefore y=\log_e\left(\frac{2}{t}\right)$.

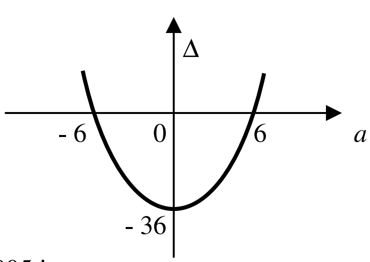
Q1bi $g(t)=(t-1)^2e^{-t}$, $t\geq 0$.
$g'(t)=2(t-1)e^{-t}-(t-1)^2e^{-t}=\left(-t^2+4t-3\right)e^{-t}$.
$\therefore b=4,\ c=-3$.

Q1bii At stationary points, $g'(t)=0$, $\therefore\left(-t^2+4t-3\right)e^{-t}=0$.
Since $e^{-t}\neq 0$, $\therefore\left(-t^2+4t-3\right)=0$, $(-t+1)(t-3)=0$,
$\therefore t=1$ and $y=0$ or $t=3$ and $y=4e^{-3}$.
Hence $p=0$, $m=3$ and $n=4e^{-3}$.

Q1biii $q(t)$ is a transformation of $g(t)$: Vertical dilation by a factor of 2 and then downward translation by 5 units. These transformations affect only the y-coordinates of points.
$(1,0)\to(1,(2\times 0-5))$, i.e. $(1,-5)$.
$\left(3,4e^{-3}\right)\to\left(3,\left(2\times 4e^{-3}-5\right)\right)$, i.e. $\left(3,\left(8e^{-3}-5\right)\right)$.

Q1ci $h(t)=\left(t^2+at+10\right)e^{-t}$;
$h'(t)=\left(-t^2+(2-a)t+(a-10)\right)e^{-t}$.
At stationary points, $h'(t)=0$.
$\therefore\left(-t^2+(2-a)t+(a-10)\right)e^{-t}=0$.
$\therefore -t^2+(2-a)t+(a-10)=0$. To have exactly one solution to this equation, $\Delta=(2-a)^2+4(a-10)=0$, i.e. $a^2-36=0$.
Hence $a=\pm 6$.

Q1cii Since $e^{-t}>0$,
for $h'(t)<0$, $-t^2+(2-a)t+(a-10)<0$.
$\therefore -t^2+(2-a)t+(a-10)$ does not cross the t-axis. Hence its $\Delta=a^2-36<0$, $\therefore -6<a<6$. Refer to the following graph.



Q2a $\Pr(X>81.80)=0.41207\approx 0.412$
$\Pr(81.80<X<90.17)=0.393409\approx 0.393$
$\Pr(X>90.17)=0.018661\approx 0.019$.

Q2b $\Pr(X\geq M)=0.90$, $\therefore \Pr(X<M)=0.10$,
$\therefore M=75.03302\approx 75.03$

Q2c Conditional probability:
$\Pr(X\geq 81.80|X<90.17)=\frac{\Pr(81.80\leq X<90.17)}{\Pr(X<90.17)}$
$=\frac{0.393409}{1-0.018661}=0.40089\approx 0.401$.

Q2d For each throw,
$E(reward)=0\times 0.5+1000\times(1-0.41207-0.5)+2000\times 0.393409$
$+10000\times 0.018661=1061.36\approx \$1060$.

Q2ei For 5 throws,
$E(totalreward)=5\times 1061.36=5306.79\approx \$5310$.

Q2eii Binomial distribution: $n=5$, $p=0.41207$,
$\Pr(X\geq 3)=1-\Pr(X\leq 2)=1-0.661501=0.338499\approx 0.338$.

Q2eiii $E(X)=np=5\times 0.41207=2.06035\approx 2.06$.

Q2eiv There must be at least one over the Olympic Record or 5 between A Standard and Olympic Record in order to earn at least \$10000.
Binomial: $n=5$, $p=0.018661$,
$\Pr(X\geq 1)=1-\Pr(X=0)=1-0.9101=0.08989$
Binomial: $n=5$, $p=0.393409$,
$\Pr(X=5)=0.0094237$
Required probability $=0.08989+0.0094237=0.0993\approx 0.099$.

Q3a Vertical distance between the mountain top and the tunnel = max value of $y=100+50=150$ metres.

Q3b Vertical distance between the valley bottom and the bridge = |min value of $y| = |-100+50|=50$ metres.

Q3ci When $y=0$, $100\cos\left[\frac{\pi(x-400)}{600}\right]+50=0$,
$\cos\left[\frac{\pi(x-400)}{600}\right]=-\frac{50}{100}=-\frac{1}{2}$,
$\frac{\pi(x-400)}{600}=\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$. $\therefore x=800$.
Length of tunnel = 800 metres.

Q3cii Length of bridge $=2\left(600-\frac{1}{2}\times 800\right)=400$ metres.

Q3d When $y = 20$, $100\cos\left[\frac{\pi(x-400)}{600}\right] + 50 = 20$,

$\cos\left[\frac{\pi(x-400)}{600}\right] = -\frac{30}{100} = -0.3$,

$\frac{\pi(x-400)}{600} = \cos^{-1}(-0.3) = -1.8755$ or 1.8755.

$\therefore x = 41.806$ or 758.192.

Length of tunnel $= 758.192 - 41.806 = 716.386 \approx 716$ metres.

Q3e Start of the dam wall $x = 800$, middle of the dam wall $x = 1000$. $Area = -2\int 100\cos\left[\frac{\pi(x-400)}{600}\right] + 50dx$

$$= -2\left[\frac{100\sin\left[\frac{\pi(x-400)}{600}\right]}{\frac{\pi}{600}} + 50x\right]_{800}^{1000}$$

$$= -2\left(\frac{60000\sin(\pi)}{\pi} + 50000 - \frac{60000\sin\left(\frac{2\pi}{3}\right)}{\pi} - 40000\right)$$

$= 13080\text{ m}^2$.

Q3fi Length of tunnel $= 800 - 2k = 2(400-k)$ metres.

Q3fii

Length of bridge $= (600 - (400-k)) \times 2 = 2(200+k)$ metres.

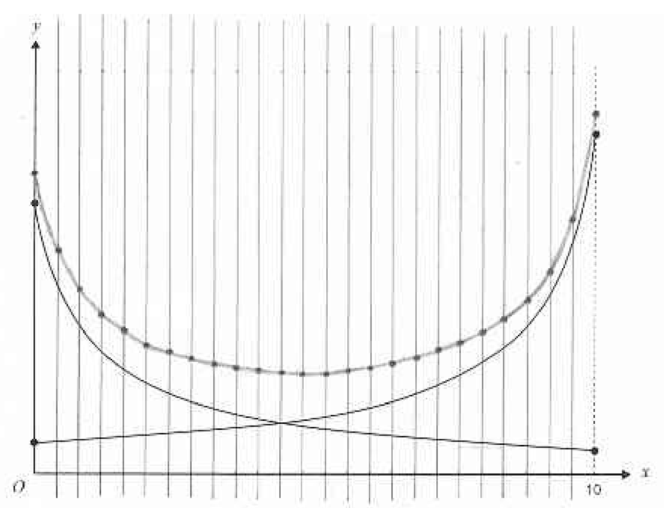
Q3fiii

$C = (2(400-k))^2 + (2(200+k))^2 = 4(400-k)^2 + 4(200+k)^2$ thousand dollars.

Q3fiv $\frac{dC}{dk} = -8(400-k) + 8(200+k) = 16(k-100)$. Cost is minimum when $16(k-100) = 0$, i.e. $k = 100$.

Q4a At $x = 3$, $y = \frac{p}{4} + \frac{q}{8} = \frac{2p+q}{8}$.

Q4b Addition of ordinates.



Q4c $y = 9(x+1)^{-1} + 4(11-x)^{-1}$,

$\frac{dy}{dx} = -9(x+1)^{-2} + 4(11-x)^{-2} = -\frac{9}{(x+1)^2} + \frac{4}{(11-x)^2}$.

At the minimum, $-\frac{9}{(x+1)^2} + \frac{4}{(11-x)^2} = 0$,

$4(x+1)^2 - 9(11-x)^2 = 0$,

$\therefore [2(x+1) - 3(11-x)][2(x+1) + 3(11-x)] = 0$,

or $(5x-31)(35-x) = 0$, where $0 \le x \le 10$.

Q4di $(5x-31)(35-x) = 0$, where $0 \le x \le 10$.

$\therefore x = \frac{31}{5} = 6.200$ and $y = \frac{9}{6.2+1} + \frac{4}{11-6.2} = 2.083$.

Q4dii $\frac{9}{x+1} + \frac{4}{11-x} < 5$, and $0 \le x \le 10$.

Use graphics calculator to sketch $y = \frac{9}{x+1} + \frac{4}{11-x}$ and $y = 5$.

Find the x-coordinates of the intersections. The second one is outside the domain. $\therefore 0.95577 < x \le 10$.

Hence length of journey $= 10 - 0.95577 = 9.04423 \approx 9.044$ km.

Q4e $\int_0^{10}\left(\frac{9}{x+1} + \frac{4}{11-x}\right)dx = [9\log_e(x+1) - 4\log_e(11-x)]_0^{10}$

$= [9\log_e 11 - 4\log_e 1 - 9\log_e 1 + 4\log_e 11] = 13\log_e 11 = 31.17$.

Total pollution is 31.17.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors