VCAA Specialist Mathematics
Sample exam 2 solutions 2006

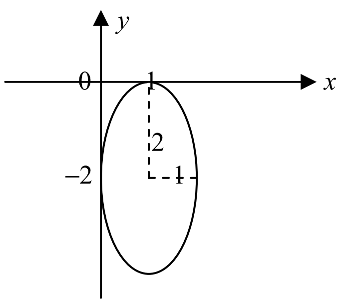
SECTION 1: Multiple-choice questions

1	2	3	4	5	6	7	8	9	10	11
E	C	E	D	B	A	C	A	D	B	E

12	13	14	15	16	17	18	19	20	21	22
D	C	E	B	D	A	B	A	B	E	C

Q1 $y=\frac{-x^2+1}{2x}=-\frac{x^2}{2x}+\frac{1}{2x}=-\frac{1}{2}x+\frac{1}{2x}$.

Q2



$\frac{(x-1)^2}{1^2}+\frac{(y+2)^2}{2^2}=1,\ \therefore (x-1)^2+\frac{(y+2)^2}{4}=1.$

Q3 A, B and D are equal to $\tan\left(\frac{\pi}{5}\right)$.

Also, $\cot\left(\frac{3\pi}{10}\right)=\frac{\cos\left(\frac{3\pi}{10}\right)}{\sin\left(\frac{3\pi}{10}\right)}=\frac{\cos\left(\frac{\pi}{2}-\frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right)}$

$=\frac{\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{5}\right)+\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{5}\right)-\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{5}\right)}=\frac{\sin\left(\frac{\pi}{5}\right)}{\cos\left(\frac{\pi}{5}\right)}=\tan\left(\frac{\pi}{5}\right)$.

Quicker to evaluate $\tan\left(\frac{\pi}{5}\right)$ and $\cot\left(\frac{3\pi}{10}\right)$ by calculator.

Q4 The period of $y=-\sec(x)$ is halved, $\therefore y=-\sec(2x)$.

Then $y=-\sec(2x)$ is translated to the right by $\frac{\pi}{4}$,

$\therefore y=-\sec\left(2\left(x-\frac{\pi}{4}\right)\right)$.

Q5 Non-real roots of a polynomial with real coefficients appear in conjugate pairs. $\therefore$ always in even number.

Q6 $w^{-1}=\frac{1}{w}=\frac{\overline{w}}{w\overline{w}}=\frac{\overline{w}}{|w|^2}=\frac{\overline{w}}{2.25}$. $\therefore w^{-1}$ has the same argument as $\overline{w}$, i.e. point *S*, but $\frac{1}{2.25}$ of $|\overline{w}|$, $\therefore$ point *P*.

Q7 Shaded region $=\left\{z:\frac{\pi}{3}<Arg(z)<\frac{\pi}{2}\right\}$.

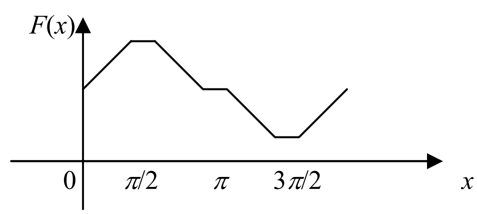
Q8 $\int_0^{\frac{\pi}{3}}\cos^2(x)\sin^3(x)dx=\int_0^{\frac{\pi}{3}}\cos^2(x)\sin^2(x)\sin(x)dx$

$=\int_0^{\frac{\pi}{3}}\cos^2(x)\left(1-\cos^2(x)\right)\sin(x)dx$.

Let $u=\cos(x)$, $\therefore \frac{du}{dx}=-\sin(x)$. When $x=0$, $u=1$, when $x=\frac{\pi}{3}$, $u=\frac{1}{2}$.

$\therefore \int_0^{\frac{\pi}{3}}\cos^2(x)\sin^3(x)dx=\int_1^{\frac{1}{2}}\left(-u^2\left(1-u^2\right)\right)du=\int_{\frac{1}{2}}^{1}u^2\left(1-u^2\right)du$.

Q9 Stationary points: $F'(x)=f(x)=0$, from graph $x=\frac{\pi}{2},\pi,\frac{3\pi}{2}$.



Q10 Differentiate $x^2+y^2=9$ implicitly, $2x+2y\frac{dy}{dx}=0$,

$\therefore \frac{dy}{dx}=-\frac{x}{y}$. In the third quadrant at $x=-1$, $y=-2\sqrt{2}$,

$\frac{dy}{dx}=-\frac{-1}{-2\sqrt{2}}=-\frac{1}{2\sqrt{2}}$.

Q11 $a<b<0$, $\int_a^b\frac{1}{x}dx=\left[\log_e|x|\right]_a^b=\log_e|b|-\log_e|a|$.

Note: $\log_e(a)$ and $\log_e(b)$ are undefined because *a* and *b* are negative values. $\therefore A,B,C$ and *D* are undefined.

Q12 $y=\sqrt{x^2-9}$, $\therefore x^2=y^2+9$. At $x=5$, $y=\sqrt{5^2-9}=4$.

Volume $=\int_0^4\left(\pi 5^2-\pi x^2\right)dy=\pi\int_0^4\left(25-\left(y^2+9\right)\right)dy$

$=\pi\int_0^4\left(16-y^2\right)dy$.

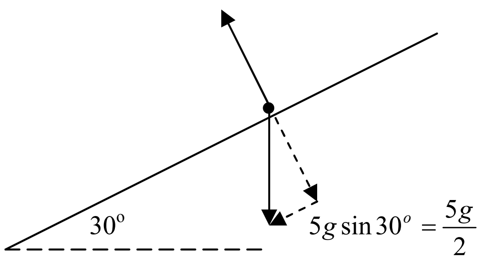
Q13 **b.**(**a** – **b**) = **b.c** = 0 because **b** $\perp$ **c**.

Q14 $\mathbf{r}(t) = 2\sin(t)\,\mathbf{i} + \cos(t)\,\mathbf{j},\ 0 \le t \le \pi$.

$\therefore x = 2\sin(t),\ y = \cos(t).\ \therefore \left(\frac{x}{2}\right)^2 + y^2 = 1$, i.e. $\frac{x^2}{4} + y^2 = 1$.

As t increases from 0 to π, x increases from 0 to 2 (when $t = \frac{\pi}{2}$) and back to 0. $\therefore 0 \le x \le 2$.

Q15



$a = \frac{R}{m} = \frac{\frac{5g}{2}}{5} = \frac{g}{2}$.

Q16 Let T be the tension in each string. $2T\cos 30^o - 10g = 0$,

$\therefore T = \frac{10g}{\sqrt{3}} = \frac{10g\sqrt{3}}{3}$.

Q17 Resultant force = **R** + **S** + **T** = 6**i** + 8**j**

Acceleration = (6**i** + 8**j**)/5, $\mathbf{a} = \frac{6}{5}\mathbf{i} + \frac{8}{5}\mathbf{j}$,

$\therefore a = \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{8}{5}\right)^2} = 2$.

Q18 Take upward direction as positive.

$u = +21,\ a = -9.8,\ t = 10,\ s = -h$. Use $s = ut + \frac{1}{2}at^2$,

$-h = 210 - 490,\ \therefore h = 280$.

Q19 The body is on the point of sliding down the plane, $\therefore$the frictional force F must be parallel and up the plane. N is the normal reaction and must be perpendicular to the plane. Weight force is Mg.

Q20 The slope field indicates that the function has an x-intercept, $\therefore$it cannot be A, C or D. The slope field is defined for $x \in R^+$, $\therefore$the function cannot be E.

Q21 Excess of water temperature over the temperature inside the refrigerator is $y - 4$. At $t = 0,\ y = 20$.

Q22 Use $y_{new} \approx y_{old} + hy'_{old}$, $y' = \cos\left(\frac{x}{2}\right),\ h = 0.2$.

$x = 0,\ y = 2$

$x = 0.2,\ y \approx 2 + 0.2\cos 0 = 2.2$

$x = 0.4,\ y \approx 2.2 + 0.2\cos(0.1)$

SECTION 2

Q1a $\mathbf{r} = (2 - 2\cos(t))\,\mathbf{i} + (1 + \sin(t))\,\mathbf{j}$, $\mathbf{s} = \sin(t)\,\mathbf{i} + 2\cos(t)\,\mathbf{j}$, $t > 0$.

$\mathbf{r.s} = (2 - 2\cos(t))\sin(t) + 2(1 + \sin(t))\cos(t) = 2(\sin(t) + \cos(t))$.

When **r** $\perp$ **s**, **r.s** = 0, $2(\sin(t) + \cos(t)) = 0,\ \sin(t) + \cos(t) = 0$,

$\therefore \tan(t) = -1,\ \therefore t = \frac{3\pi}{4}$ is the first time.

Q1b $x = 2 - 2\cos(t),\ y = 1 + \sin(t)$,

$\therefore \cos(t) = -\frac{x-2}{2},\ \sin(t) = y - 1$.

Hence $\left(-\frac{x-2}{2}\right)^2 + (y-1)^2 = 1$, i.e. $\frac{(x-2)^2}{4} + (y-1)^2 = 1$, where $0 \le x \le 4$.

Q1c For R: Velocity = d**r**/dt $= 2\sin(t)\,\mathbf{i} + \cos(t)\,\mathbf{j}$.

Speed $= \sqrt{(2\sin(t))^2 + \cos^2(t)} = \sqrt{4\sin^2(t) + \cos^2(t)}$.

For S: Velocity = d**s**/dt $= \cos(t)\,\mathbf{i} - 2\sin(t)\,\mathbf{j}$.

Speed $= \sqrt{\cos^2(t) + (-2\sin(t))^2} = \sqrt{4\sin^2(t) + \cos^2(t)}$.

$\therefore$ same speed at any given time.

Find t such that $2\sin(t)\,\mathbf{i} + \cos(t)\,\mathbf{j} = \cos(t)\,\mathbf{i} - 2\sin(t)\,\mathbf{j}$,

i.e. $2\sin(t) = \cos(t)$ and $\cos(t) = -2\sin(t)$,

i.e. $\tan(t) = \frac{1}{2}$ and $\tan(t) = -\frac{1}{2}$. No t can satisfy both equations simultaneously, $\therefore$R and S never have the same velocity.

Q2ai $u = 0,\ s = +400,\ t = 8,\ a = ?$ Use $s = ut + \frac{1}{2}at^2$,

$400 = \frac{1}{2}a8^2,\ a = 12.5$

Q2aii $u = 0,\ s = +400,\ t = 8,\ v = ?$ Use $s = \frac{1}{2}(u+v)t$,

$400 = \frac{1}{2}v8,\ v = 100$

Q2bi $a = \frac{R}{m} = \frac{-5000 - 0.5v^2}{400}$. $a = \frac{-5000 - 0.5v^2}{400}$ is the equation of motion.

Q2bii Let $a = v\frac{dv}{dx}$,

$\therefore v\frac{dv}{dx} = \frac{-5000 - 0.5v^2}{400} = \frac{-10000 - v^2}{800}$.

$\therefore \frac{dv}{dx} = \frac{-(10^4 + v^2)}{800v}$.

Q2biii $\frac{dx}{dv} = -\frac{800v}{10^4 + v^2}$, $x = \int -\frac{800v}{10^4 + v^2}\, dv$.

Let variable $u = 10^4 + v^2$, $\frac{du}{dv} = 2v$,

$\therefore x = \int -\frac{400}{u}\frac{du}{dv}\, dv = \int -\frac{400}{u}\, du = -400 \log_e |u| + C$

$= -400 \log_e \left(10^4 + v^2\right) + C$.

At $x = 0$, $v = 100$, $\therefore C = 400 \log_e 2\times 10^4$.

Hence

$x = 400 \log_e 2\times 10^4 - 400\log_e\left(10^4 + v^2\right) = 400 \log_e\left(\frac{2\times 10^4}{10^4 + v^2}\right)$.

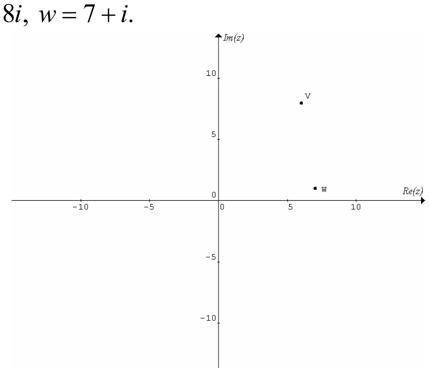
When $v = 0$, $x = 400\log_e 2 = 277$, $\therefore$distance = 277 m.

Q2c Let $a = \frac{dv}{dt}$, $\therefore \frac{dv}{dt} = -\frac{10^4 + v^2}{800}$, $\frac{dt}{dv} = -\frac{800}{10^4 + v^2}$,

$\therefore T = \int_{100}^{0} -\frac{800}{10^4 + v^2}\, dv = \int_0^{100} \frac{800}{10^4 + v^2}\, dv.$

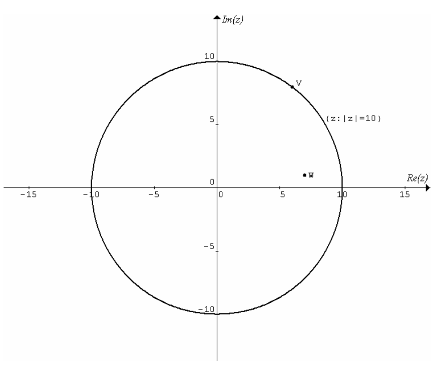
$\therefore T = \int_0^{100} \frac{8\times 10^2}{10^4 + v^2}\, dv = 8\left[\tan^{-1}\left(\frac{v}{100}\right)\right]_0^{100} = 8\tan^{-1}(1) = 2\pi = 6.28$

Q3a $v = 6 + 8i$, $w = 7 + i$.



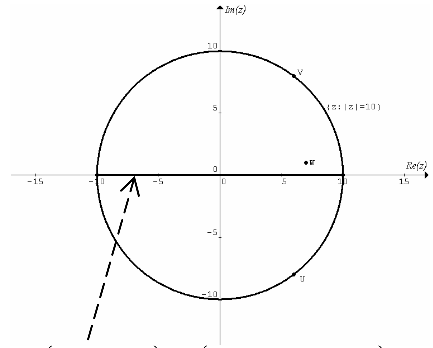
Q3bi $|v| = \sqrt{6^2 + 8^2} = 10$, $\therefore v \in S$

Q3bii



Q3c $u + i\overline{w} = \overline{w}$, $\therefore u = \overline{w} - i\overline{w} = \overline{w}(1 - i) = (7 - i)(1 - i) = 6 - 8i$

Q3d $\{z : |z| \le 10\}$ is the circle and its interior. $\{z : |z - u| = |z - v|\}$ is a straight line equidistant from *u* and *v*.



$\{z : |z| \le 10\} \cap \{z : |z - u| = |z - v|\}$

Q3e Let **i** be a unit vector in the direction of $\mathrm{Re}(z)$ axis, and **j** a unit vector in the direction of $\mathrm{Im}(z)$ axis.

$\overrightarrow{OW} = 7\,\mathbf{i} + \mathbf{j}$, $\overrightarrow{WV} = \overrightarrow{OV} - \overrightarrow{OW} = (6\,\mathbf{i} + 8\,\mathbf{j}) - (7\,\mathbf{i} + \mathbf{j}) = -\mathbf{i} + 7\mathbf{j}$.

$\therefore \overrightarrow{OW} \bullet \overrightarrow{WV} = -7 + 7 = 0$, $\therefore \overrightarrow{OW} \perp \overrightarrow{WV}$, i.e. $\angle OWV = 90^o$.

Q4a *f* is defined when $1 - x^2 > 0$ and $x \ge 0$, i.e. $-1 < x < 1$ and $x \ge 0$, $\therefore$the largest domain is $[0,1)$.

Q4b When $x = 0.5$,

$y = 2(0.5)^{\frac{1}{2}}\left(1 - 0.5^2\right)^{\frac{1}{4}} + \left(1 - 0.5^2\right)^{-\frac{1}{4}} = 2.391$

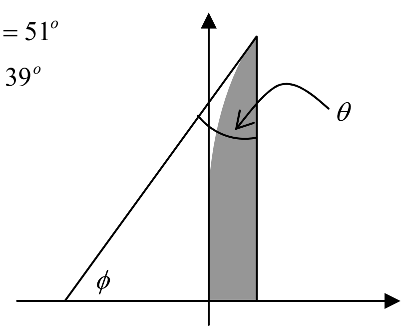
$\therefore D = 2\times 2.39 = 4.78$

Q4c $f'(x) = -x^{\frac{3}{2}}\left(1 - x^2\right)^{-\frac{3}{4}} + x^{-\frac{1}{2}}\left(1 - x^2\right)^{\frac{1}{4}} + \frac{1}{2}x\left(1 - x^2\right)^{-\frac{5}{4}}$,

$\therefore f'(0.5) = 1.2356$.

$\tan\phi = 1.2356$, $\phi = 51^o$

$\therefore \theta = 90^o - 51^o = 39^o$



Q4di The third term is $2\left[2x^{\frac{1}{2}}\left(1 - x^2\right)^{\frac{1}{4}}\right]\left[\frac{1}{\left(1 - x^2\right)^{\frac{1}{4}}}\right] = 4\sqrt{x}$.

Q4dii $V = \int_0^{0.5} \pi y^2\, dx = \int_0^{0.5} \pi\left(4x\sqrt{1 - x^2} + 4\sqrt{x} + \frac{1}{\sqrt{1 - x^2}}\right) dx$

Q4diii $V = \pi\left[-\frac{4\left(1 - x^2\right)^{\frac{3}{2}}}{3} + \frac{8x^{\frac{3}{2}}}{3} + Sin^{-1}x\right]_0^{0.5} = 6.1\ \mathrm{m}^3$.

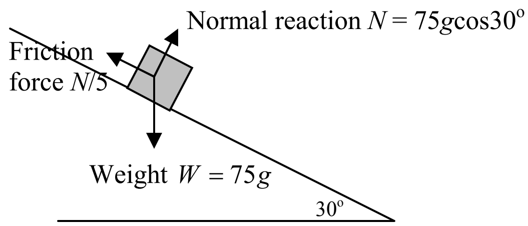
$\therefore$7 packages are required.

Note: for $\int 4x\sqrt{1 - x^2}\, dx$, let $u = 1 - x^2$, $\frac{du}{dx} = -2x$,

$\therefore \int 4x\sqrt{1 - x^2}\, dx = \int -2\sqrt{u}\frac{du}{dx}\, dx = \int -2\sqrt{u}\, du$

$= -\frac{4u^{\frac{3}{2}}}{3} + C = -\frac{4\left(1 - x^2\right)^{\frac{3}{2}}}{3} + C$.

Q5a



Q5b Along the slide,

$$a = \frac{R}{m} = \frac{75g\sin 30^o - \frac{N}{5}}{75} = \frac{\frac{75g}{2} - \frac{75g\sqrt{3}}{10}}{75} = \frac{g}{2}\left(1 - \frac{\sqrt{3}}{5}\right).$$

Q5ci Motion under constant acceleration:

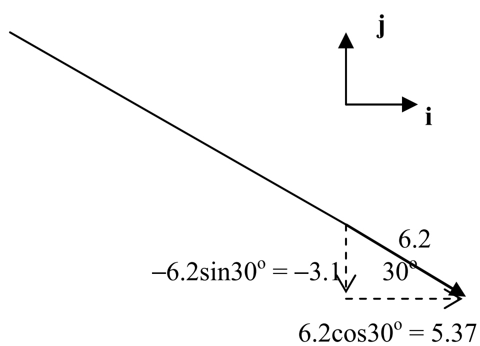
$u = 0,\ s = 6,\ a = \frac{g}{2}\left(1 - \frac{\sqrt{3}}{5}\right) = 3.2026,\ t = ?$ Use $s = ut + \frac{1}{2}at^2$,

$t = 1.94$.

Q5cii $u = 0,\ s = 6,\ a = 3.2026,\ v = ?$ Use $v^2 = u^2 + 2as$,

$\therefore v = \sqrt{2as} = 6.2$.

Q5d



Consider the **j** component:

$u = -3.1,\ s = -2,\ a = -9.8,\ t = ?$ Use $s = ut + \frac{1}{2}at^2$,

$-2 = -3.1t - 4.9t^2,\ 4.9t^2 + 3.1t - 2 = 0,\ t = 0.40$.

Consider the **i** component:

$u = +5.37,\ t = 0.40,\ s = ?$ Use $s = ut$ (Note: $a = 0$),

$s = 2.1$.

Distance = 2.1 m.

Q5e Landing velocity, **i** component is 5.37.

j component: $u = -3.1,\ s = -2,\ a = -9.8,\ v = ?$

Use $v^2 = u^2 + 2as,\ v^2 = (-3.1)^2 + 2(-9.8)(-2),\ \therefore v = -6.98$.

Landing velocity $\mathbf{v} = 5.37\,\mathbf{i} - 6.98\,\mathbf{j}$.

Landing momentum

$\mathbf{p} = m\mathbf{v} = 75(5.37\,\mathbf{i} - 6.98\,\mathbf{j}) = 402.75\,\mathbf{i} - 523.5\,\mathbf{j}$.

$|\mathbf{p}| = \sqrt{402.75^2 + (-523.5)^2} = 6.60 \times 10^2$

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