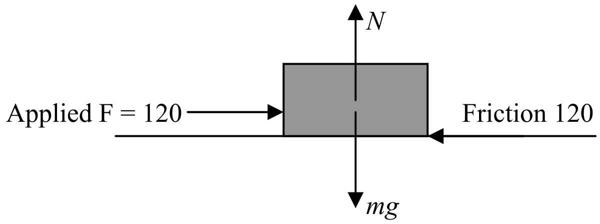
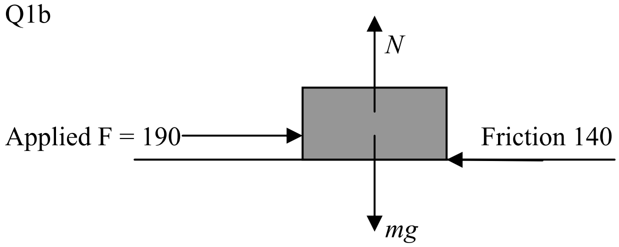
VCAA Specialist Mathematics
Sample exam 1 solutions 2006

Q1a Limiting friction $= \mu N = \mu mg = \frac{1}{7}(100)(9.8) = 140$ newtons.



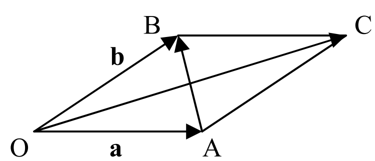
The applied force 120 N is lower than the limiting friction 140 N $\therefore$ friction force is 120 N in the opposite direction to the applied force. Hence the resultant force is zero. Since the crate is at rest, it will remain at rest according to Newton's first law.

Q1b



$a = \frac{R}{m} = \frac{190-140}{100} = 0.5$ ms^{-2}.

Q2



$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{OC} = \mathbf{a} + \mathbf{b}$.

Since $\overrightarrow{AB} \perp \overrightarrow{OC}$, $\therefore \overrightarrow{AB} \bullet \overrightarrow{OC} = (\mathbf{b} - \mathbf{a}).(\mathbf{a} + \mathbf{b}) = 0$.

Expand to obtain $\mathbf{b.a} + \mathbf{b.b} - \mathbf{a.a} - \mathbf{a.b} = 0$,

$\therefore \mathbf{b.b} - \mathbf{a.a} = 0$ or $b^2 - a^2 = 0$.

$\therefore a = b$. Hence the parallelogram OACB is a rhombus.

Q3a $f(x) = Tan^{-1}(x) + xTan^{-1}(x) = (1+x)Tan^{-1}(x)$,

$f'(x) = Tan^{-1}(x) + (1+x)\left(\frac{1}{x^2+1}\right) = Tan^{-1}(x) + \frac{x+1}{x^2+1}$.

Q3b At the point of inflection, $f''(x) = 0$.

$f''(x) = \frac{1}{x^2+1} + \frac{(x^2+1) - 2x(x+1)}{(x^2+1)^2} = \frac{x^2+1+x^2+1-2x^2-2x}{(x^2+1)^2} = 0$,

$\therefore 2 - 2x = 0,\ x = 1,\ y = (1+1)Tan^{-1}(1) = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$. $\therefore \left(1, \frac{\pi}{2}\right)$.

Q4a $2y - xy^2 = 8$, $\frac{d}{dx}(2y - xy^2) = \frac{d}{dx}(8)$,

$2\frac{dy}{dx} - \left(y^2 + x(2y)\frac{dy}{dx}\right) = 0$. $\therefore 2\frac{dy}{dx} - 2xy\frac{dy}{dx} = y^2$,

$\therefore 2(1-xy)\frac{dy}{dx} = y^2$. Hence $\frac{dy}{dx} = \frac{y^2}{2(1-xy)}$.

Q4b When $y = 2$, $2(2) - x(2^2) = 8$, $x = -1$.

$\therefore \frac{dy}{dx} = \frac{2^2}{2(1-(-1)2)} = \frac{2}{3}$.

Q5a $z^3 - 2z^2 + 9z - 18 = (3i)^3 - 2(3i)^2 + 9(3i) - 18$

$= -27i + 18 + 27i - 18 = 0$.

$\therefore z = 3i$ is a solution of $z^3 - 2z^2 + 9z - 18 = 0$.

Q5b Hence $z - 3i$ is a factor of $z^3 - 2z^2 + 9z - 18$. Since $z^3 - 2z^2 + 9z - 18$ has real coefficients, $\therefore z + 3i$ is also a factor.

$\therefore z^3 - 2z^2 + 9z - 18 = (z-3i)(z+3i)(z-p)$.

$\therefore -18 = (-3i)(3i)(-p)$, $\therefore p = 2$.

$\therefore$ the solutions are $3i, -3i, 2$.

Q6a Let $u = \sqrt{3x}$,

$\frac{d}{dx}\left(\cos^{-1}(u)\right) = \frac{d}{du}\left(\cos^{-1}(u)\right)\frac{du}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{\sqrt{3}}{2\sqrt{x}}$

$= \frac{-\sqrt{3}}{2\sqrt{x}\sqrt{1-3x}} = \frac{-\sqrt{3}}{2\sqrt{x(1-3x)}} = \frac{-\sqrt{3}}{2\sqrt{x-3x^2}}$.

Q6b $\frac{d}{dx}\left(\cos^{-1}\left(\sqrt{3x}\right)\right) = \frac{-\sqrt{3}}{2\sqrt{x-3x^2}}$,

$\therefore \int_{\frac{1}{6}}^{\frac{1}{4}} \frac{-\sqrt{3}}{2\sqrt{x-3x^2}}\,dx = \left[\cos^{-1}\left(\sqrt{3x}\right)\right]_{\frac{1}{6}}^{\frac{1}{4}} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$,

$\therefore -\frac{\sqrt{3}}{2}\int_{\frac{1}{6}}^{\frac{1}{4}} \frac{1}{\sqrt{x-3x^2}}\,dx = \frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12}$,

$\therefore \int_{\frac{1}{6}}^{\frac{1}{4}} \frac{1}{\sqrt{x-3x^2}}\,dx = \frac{\sqrt{3}\pi}{18}$.

Q7

$y = \int \sin^2(x)\cos^2(x)\sin(x)dx = \int (1-\cos^2(x))\cos^2(x)\sin(x)dx$

$= \int (\cos^2(x) - \cos^4(x))\sin(x)dx$.

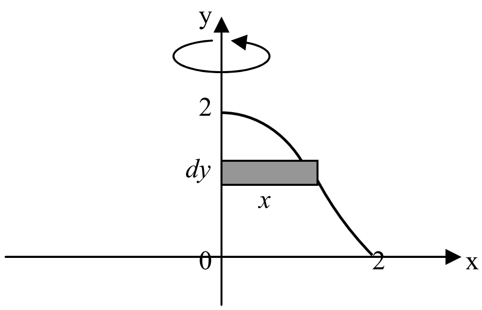
Let $u = \cos(x)$, $\frac{du}{dx} = -\sin(x)$,

$\therefore y = \int -(\cos^2(x) - \cos^4(x))\frac{du}{dx}dx$, $\therefore y = \int (u^4 - u^2)du$,

$y = \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C$. Given $y(0) = 0$, $\therefore 0 = \frac{1}{5} - \frac{1}{3} + C$,

$\therefore C = \frac{2}{15}$. Hence $y = \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + \frac{2}{15}$.

Q8



$y = \frac{16}{x^2+4} - 2$, $\therefore x^2 = \frac{16}{y+2} - 4$.

Volume $= \int_0^2 \pi x^2 dy = \int_0^2 \pi\left(\frac{16}{y+2} - 4\right)dy = \pi\left[16\log_e|y+2| - 4y\right]_0^2$

.For $-2 \le x \le 2$, $y \ge 0$, $\therefore y + 2 > 0$,

$\therefore$ volume $= \pi\left[16\log_e(y+2) - 4y\right]_0^2$

$= \pi(16\log_e 4 - 8) - \pi 16\log_e 2 = 8\pi(\log_e 4 - 1)$.

Q9a The discriminant of $x^2 + 4x + 5$ is $\Delta = 4^2 - 4(1)(5) = -4$,

$\therefore x^2 + 4x + 5 > 0$ for all *x*. Hence $y = \frac{3}{x^2+4x+5}$ is defined for all *x*.

Q9b $\int \frac{3}{x^2+4x+5}dx = \int \frac{3}{1+(x+2)^2}dx = 3\tan^{-1}(x+2) + c$.

$\therefore A = 3$, $B = 2$.

Q9c The turning point is $(-2,1)$. $\therefore a = -2$.

Area $= \int_{-2}^{\sqrt{3}-2} \frac{3}{x^2+4x+5}dx = \left[3\tan^{-1}(x+2)\right]_{-2}^{\sqrt{3}-2}$

$= 3\tan^{-1}(\sqrt{3}) - 3\tan^{-1}(0) = \pi$.

Q9d Consider $f(x) = 5\tan^{-1}(x+7)$, it has a range of $\left(-\frac{5\pi}{2}, \frac{5\pi}{2}\right)$.

$\therefore -\frac{5\pi}{2} < 5\tan^{-1}(x+7)$ for all *x*, $\therefore 0 < 5\tan^{-1}(x+7) + \frac{5\pi}{2}$.

Hence $c > \frac{5\pi}{2}$.

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