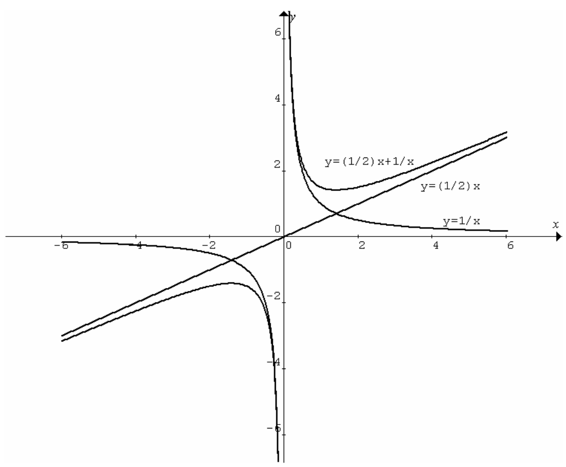
Functions, relations and graphs

In graph sketching, one must find x and y-intercepts, show asymptotic behaviour and, determine location of stationary points.

Graphs of functions defined by $f(x) = ax^m + \dfrac{b}{x^n}$ for $m, n = 1,2$

Use the method of addition of ordinates to sketch this type of functions.

Example 1 Sketch $y = \frac{1}{2}x + \frac{1}{x}$.



Clearly the function has no axis intercepts. It shows asymptotic behaviour:

As $x \to 0^-$, $y \to -\infty$. As $x \to 0^+$, $y \to +\infty$.

$\therefore x = 0$ is an asymptote (vertical) of the function.

As $x \to -\infty$, $y \to \frac{1}{2}x$ (from below).

As $x \to +\infty$, $y \to \frac{1}{2}x$ (from above).

$\therefore y = \frac{1}{2}x$ is an asymptote (oblique) of the function.

Use calculus to find the coordinates of the stationary points:

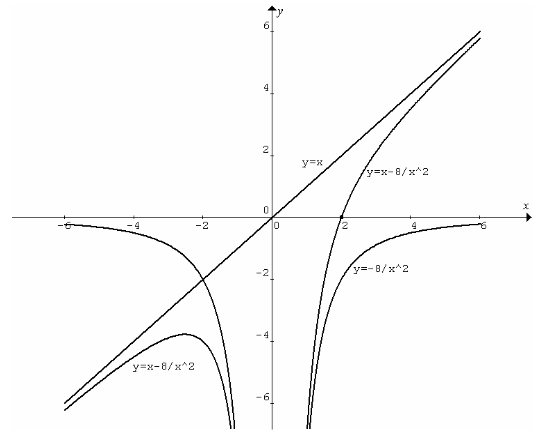
$y = \frac{1}{2}x + \frac{1}{x}$, $\frac{dy}{dx} = \frac{1}{2} - \frac{1}{x^2} = 0$, $\therefore x = \pm\sqrt{2}$.

At $x = -\sqrt{2}$, $y = \frac{1}{2}\left(-\sqrt{2}\right) + \frac{1}{-\sqrt{2}} = -\sqrt{2}$. At $x = \sqrt{2}$,

$y = \frac{1}{2}\left(\sqrt{2}\right) + \frac{1}{\sqrt{2}} = \sqrt{2}$.

The stationary points are $\left(-\sqrt{2}, -\sqrt{2}\right)$, $\left(\sqrt{2}, \sqrt{2}\right)$.

Example 2 Sketch $y = x - \frac{8}{x^2}$.



x-intercept: Let $y = 0$, $x - \frac{8}{x^2} = 0$, $x^3 = 8$, $\therefore x = 2$.

Asymptotic behaviour:

As $x \to 0^-$, $y \to -\infty$. As $x \to 0^+$, $y \to -\infty$.

$\therefore x = 0$ is an asymptote of the function.

As $x \to -\infty$, $y \to x$ (from below).

As $x \to +\infty$, $y \to x$ (from below).

$\therefore y = x$ is an asymptote of the function.

Stationary points:

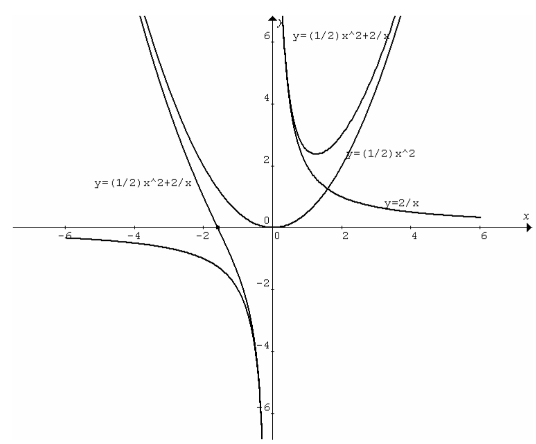
$y = x - \frac{8}{x^2}$, $\frac{dy}{dx} = 1 + \frac{16}{x^3} = 0$, $\therefore x = \sqrt[3]{-2^4} = -2^{\frac{4}{3}}$.

At $x = -2^{\frac{4}{3}} = -2.52$,

$$y = -2^{\frac{4}{3}} - \frac{8}{\left(-2^{\frac{4}{3}}\right)^2} = -2^{\frac{4}{3}} - 2^{\frac{1}{3}} = -3.78.$$

The stationary point is $(-2.52, -3.78)$.

Example 3 Sketch $y = \frac{1}{2}x^2 + \frac{2}{x}$.



x-intercept: Let $y=0$, $\frac{1}{2}x^2+\frac{2}{x}=0$, $x^3=-4$, $\therefore x=-1.59$.

Asymptotic behaviour:

As $x\to 0^-$, $y\to-\infty$. As $x\to 0^+$, $y\to+\infty$.

$\therefore x=0$ is an asymptote of the function.

As $x\to-\infty$, $y\to\frac{1}{2}x^2$ (from below).

As $x\to+\infty$, $y\to\frac{1}{2}x^2$ (from above).

$\therefore y=\frac{1}{2}x^2$ is an asymptote of the function.

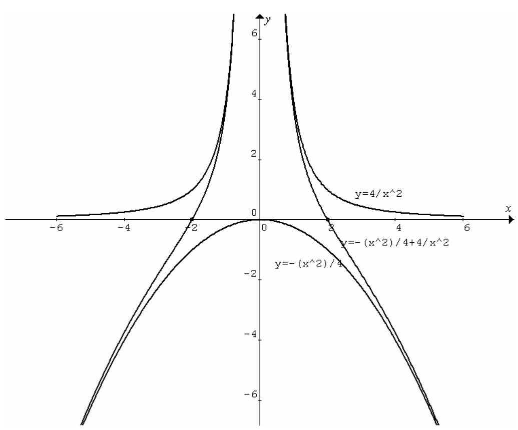
Stationary points:

$y=\frac{1}{2}x^2+\frac{2}{x}$, $\frac{dy}{dx}=x-\frac{2}{x^2}=0$, $\therefore x=\sqrt[3]{2}=2^{\frac{1}{3}}$.

At $x=2^{\frac{1}{3}}=1.26$, $y=\frac{1}{2}\left(2^{\frac{1}{3}}\right)^2+\frac{2}{2^{\frac{1}{3}}}=2^{-\frac{1}{3}}+2^{\frac{2}{3}}=2.38$.

The stationary point is $(1.26,2.38)$.

Example 4 Sketch $y=-\frac{x^2}{4}+\frac{4}{x^2}$.



x-intercept: Let $y=0$, $-\frac{x^2}{4}+\frac{4}{x^2}=0$, $x^4=16$, $\therefore x=\pm 2$.

Asymptotic behaviour:

As $x\to 0^-$, $y\to+\infty$. As $x\to 0^+$, $y\to+\infty$.

$\therefore x=0$ is an asymptote of the function.

As $x\to-\infty$, $y\to-\frac{x^2}{4}$ (from above).

As $x\to+\infty$, $y\to-\frac{x^2}{4}$ (from above).

$\therefore y=-\frac{1}{4}x^2$ is an asymptote of the function.

The function has no stationary points.

Graphs of functions defined by $f(x)=\frac{1}{ax^2+bx+c}$

The sketching method of $f(x)=\frac{1}{ax^2+bx+c}$ depends on the linear factors of ax^2+bx+c.

Case 1 ax^2+bx+c can be factorised, i.e. the discriminant $b^2-4ac>0$.

$\therefore f(x)=\frac{1}{a(x-p)(x-q)}$.

There are two vertical asymptotes, $x=p$ and $x=q$.

Example 1

Sketch the graph of the function $f(x)=\frac{1}{2x^2-x-1}$.

Check: $b^2-4ac>0$. Factorise the denominator:

$f(x)=\frac{1}{2x^2-x-1}=\frac{1}{2\left(x+\frac{1}{2}\right)(x-1)}$.

Asymptotic behaviour:

As $x\to-\frac{1}{2}$ (from the left), $y\to+\infty$.

As $x\to-\frac{1}{2}$ (from the right), $y\to-\infty$.

$\therefore x=-\frac{1}{2}$ is an asymptote (vertical).

As $x\to 1$ (from the left), $y\to-\infty$.

As $x\to 1$ (from the right), $y\to+\infty$.

$\therefore x=1$ is an asymptote (vertical).

As $x\to-\infty$, $y\to 0$ (from above).

As $x\to+\infty$, $y\to 0$ (from above).

$\therefore y=0$ is an asymptote (horizontal).

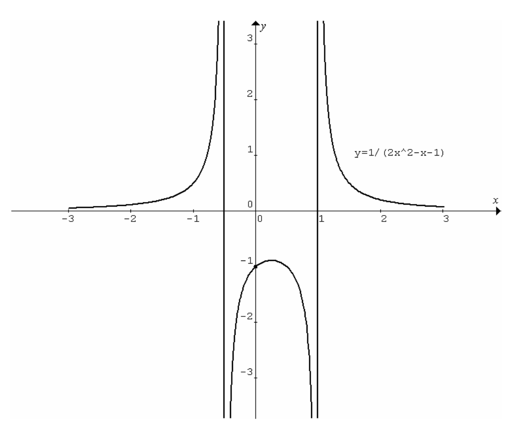
y-intercept: Let $x=0$, $y=f(0)=-1$.

The function has no x-intercepts.

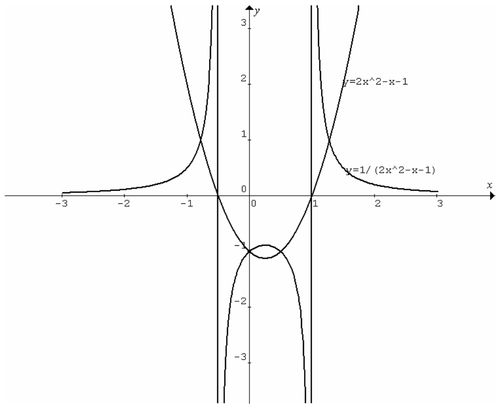
Stationary points: $y=\frac{1}{2x^2-x-1}$,

$\frac{dy}{dx}=-\frac{4x-1}{\left(2x^2-x-1\right)^2}=0$, $\therefore x=\frac{1}{4}$; $y=-\frac{8}{9}$.

Stationary point is $\left(\frac{1}{4},-\frac{8}{9}\right)$.



Alternatively, sketch the graph of $y = 2x^2 - x - 1$ first and then the reciprocal $y = \dfrac{1}{2x^2 - x - 1}$.



Case 2 $ax^2 + bx + c$ has the discriminant $b^2 - 4ac = 0$. It has two linear factors and they are the same.

$\therefore f(x) = \dfrac{1}{a(x-d)^2}$, and it is the transformation (dilation and translation) of $f(x) = \dfrac{1}{x^2}$.

Example 1 Sketch $y = \dfrac{1}{2x^2 - 4x + 2}$.

Check: $b^2 - 4ac = 0$. Factorise the denominator:

$y = \dfrac{1}{2(x-1)^2}$.

As $x \to 1$ (from the left), $y \to +\infty$.

As $x \to 1$ (from the right), $y \to +\infty$.

$\therefore x = 1$ is an asymptote (vertical).

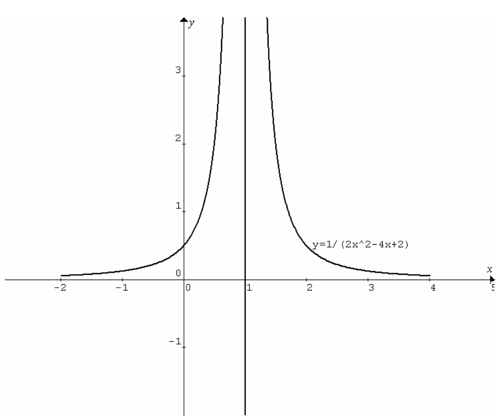
As $x \to -\infty$, $y \to 0$ (from above).

As $x \to +\infty$, $y \to 0$ (from above).

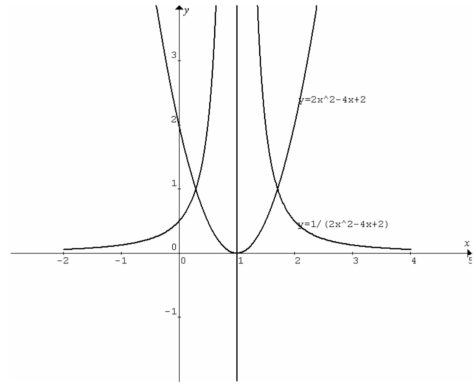
$\therefore y = 0$ is an asymptote (horizontal).

y-intercept: Let $x = 0$, $y = \dfrac{1}{2}$.

The function has no x-intercepts and stationary points.



Alternatively, sketch the graph of $y = 2x^2 - 4x + 2$ first and then the reciprocal $y = \dfrac{1}{2x^2 - 4x + 2}$.



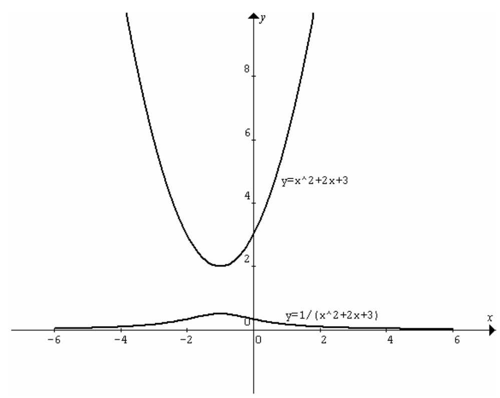
Case 3 $ax^2 + bx + c$ cannot be factorised, i.e. the discriminant $b^2 - 4ac < 0$.

In this case, the quadratic is never zero and therefore, $f(x)$ has no vertical asymptotes.

The best way to sketch $f(x)$ is to sketch the quadratic first and then its reciprocal.

Example 1 Sketch $f(x) = \dfrac{1}{x^2 + 2x + 3}$.

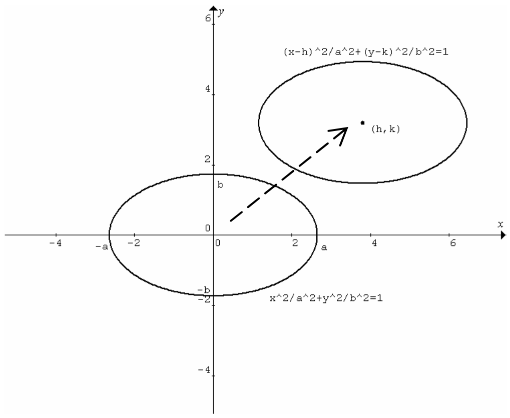
Check: $b^2 - 4ac < 0$. No linear factors.



Graphs of ellipses

The general equation of an ellipse in Cartesian form with the centre at the origin is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $\pm a$ are the x-intercepts and $\pm b$ are the y-intercepts.

If the ellipse is translated so that its centre is at (h, k), the general equation becomes $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $[-a+h, a+h]$ is the domain and $[-b+k, b+k]$ the range of the relation.

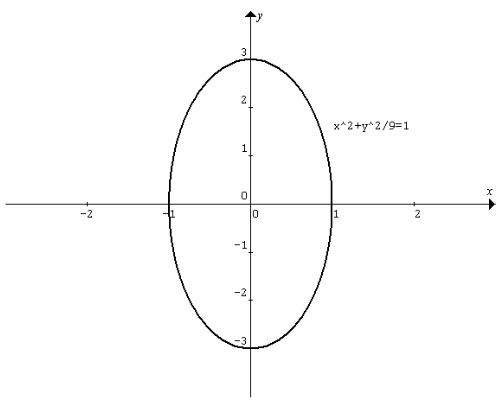


Example 1 Sketch the graphs of

a) $x^2+\dfrac{y^2}{9}=1$ b) $(x+1)^2+\dfrac{(y-2)^2}{9}=1$.

Show the x, y intercepts, the domains and ranges.

a) x-intercepts: $x=\pm 1$; y-intercepts: $y=\pm 3$



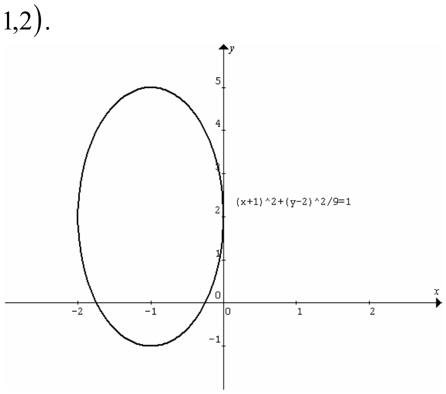
Domain: $[-1,1]$; range: $[-3,3]$.

b) x-intercepts: Let $y=0$, $(x+1)^2+\dfrac{4}{9}=1$, $(x+1)^2=\dfrac{5}{9}$,

$x+1=\pm\dfrac{\sqrt{5}}{3}$, $x=-1\pm\dfrac{\sqrt{5}}{3}$.

y-intercepts: Let $x=0$, $1+\dfrac{(y-2)^2}{9}=1$, $y=2$.

Centre: $(-1,2)$.



Domain: $[-2,0]$; range: $[-1,5]$.

Graphs of hyperbolas

The general equation of a hyperbola with its centre at the origin is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ or $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=-1$, where $\pm a$ are the x-intercepts for the first equation and $\pm b$ are the y-intercepts for the second. Both relations have two oblique asymptotes given by $y=\pm\dfrac{b}{a}x$.

The domain for the first is $R\backslash(-a, a)$ and the range is R.
The domain for the second is R and the range is $R\backslash(-b, b)$.

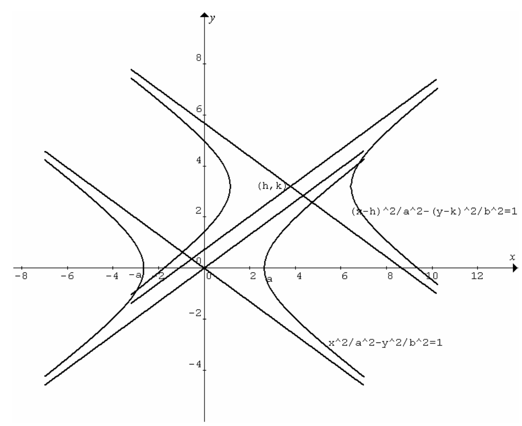
If the hyperbola is translated so that its centre is at (h, k), the general equation becomes

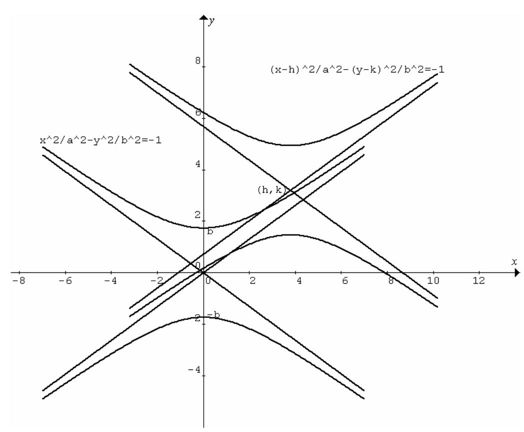
$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \text{ or } \frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=-1.$$

The two oblique asymptotes for both relations are

$y-k=\pm\dfrac{b}{a}(x-h)$.

The domain for the first is $R\backslash(-a+h, a+h)$ and the range is R.
The domain for the second is R and the range is $R\backslash(-b+k, b+k)$.





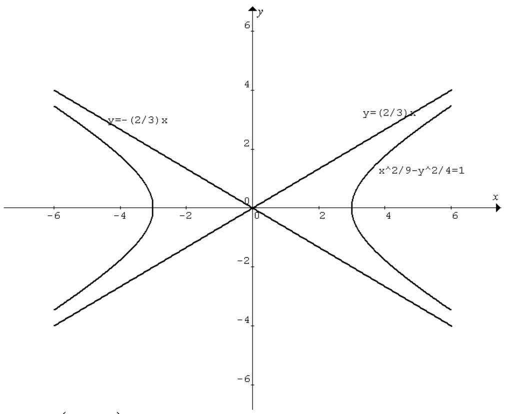
Example 1 Sketch the graphs of a) $\frac{x^2}{9}-\frac{y^2}{4}=1$ and

b) $\frac{(x+2)^2}{9}-\frac{y^2}{4}=-1$.

State the x and y intercepts, domains and ranges, and equations of the asymptotes.

a) x-intercepts: $x=\pm 3$; no y-intercepts.

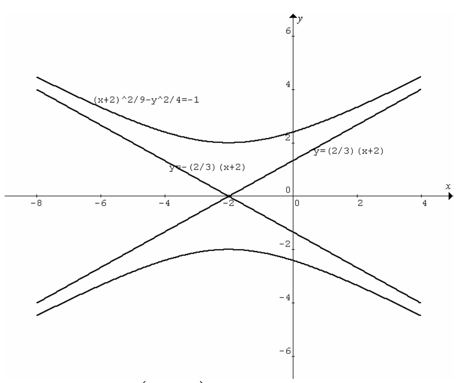
Equations of asymptotes: $y=\pm\frac{2}{3}x$.



Domain: $R\setminus(-3,3)$; range: R.

b) y-intercepts: Let $x=0$, $\frac{4}{9}-\frac{y^2}{4}=-1$, $y=\pm\frac{2\sqrt{13}}{3}$; no x-intercepts.

Equations of asymptotes: $y=\pm\frac{2}{3}(x+2)$.



Domain: R; range: $R\setminus(-2,2)$.

Reciprocal trigonometric functions

The cosecant function of x, cosec x, is defined as the reciprocal of the sine function, sin x, i.e.

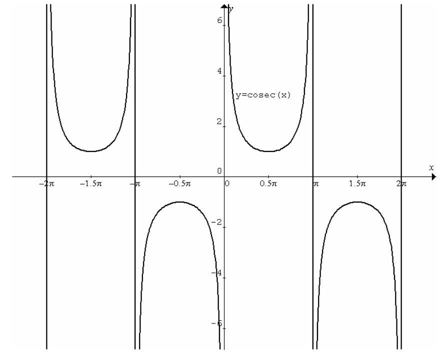
$$\operatorname{cosec} x=\frac{1}{\sin x} .$$

The other reciprocal functions are defined as

$$\sec x=\frac{1}{\cos x} \quad \text{and} \quad \cot x=\frac{1}{\tan x} .$$

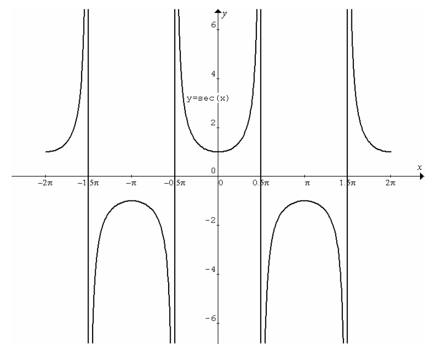
Graphs of cosec x, sec x and cot x

$y=\operatorname{cosec}(x)$



Asymptotes: $x=n\pi$, where $n=0,\pm1,\pm2,...$
Domain: $R\setminus\{x: x=n\pi, n=0,\pm1,\pm2,...\}$;
range: $(-\infty,-1]\cup[1,\infty)$.
Period: 2π .

$y=\sec(x)$

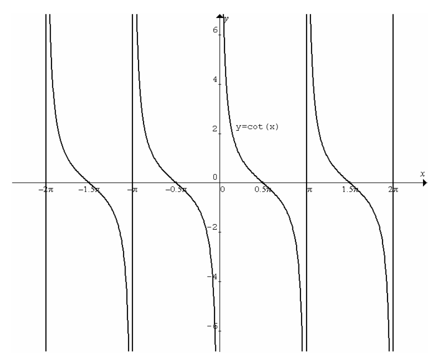


Asymptotes: $x=n\pi$, where $n=\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2}...$

Domain: $R\setminus\left\{x: x=n\pi, n=\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2}...\right\}$;

range: $(-\infty,-1]\cup[1,\infty)$.
Period: 2π .

$y=\cot(x)$

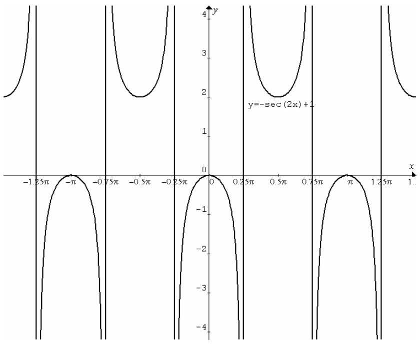


Asymptotes: $x=n\pi$, where $n=0,\pm1,\pm2,...$
Domain: $R\setminus\{x: x=n\pi, n=0,\pm1,\pm2,...\}$; range: R .
Period: π .

Transformations of cosec x, sec x and cot x

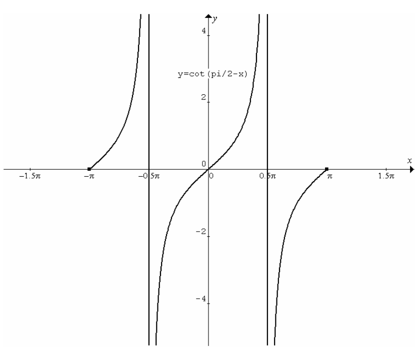
The transformations-dilation, reflection and translation are applicable to all functions. Always carry out translation last.

Example 1 Sketch $y = -\sec(2x) + 1$.
The graph of $y = \sec(x)$ is reflected in the x-axis, dilated horizontally by a factor of $\frac{1}{2}$, and translated upwards by a unit.



Asymptotes: $x = n\pi$, where $n = \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{5}{4}...$
Domain: $R \setminus \left\{x : x = n\pi, n = \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{5}{4}...\right\}$;
range: $(-\infty, 0] \cup [2, \infty)$.
Period: π.

Example 2 Sketch $y = \cot\left(\frac{\pi}{2} - x\right)$ and $-\pi \le x \le \pi$.
Rewrite: $y = \cot\left(\frac{\pi}{2} - x\right) = \cot\left[-\left(x - \frac{\pi}{2}\right)\right]$.
The graph of $y = \cot(x)$ is reflected in the y-axis, and then translated to the right by $\frac{\pi}{2}$.
The graph is restricted to $-\pi \le x \le \pi$.



Asymptotes: $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.
Domain: $[-\pi, \pi] \setminus \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$; range: R.
Period: π.

Identities

An identity is an equality which is always true as long as it is defined.

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$
$$\sin^2 x + \cos^2 x = 1$$
$$\cos^2 x = 1 - \sin^2 x$$
$$\sin^2 x = 1 - \cos^2 x$$
$$\sec^2 x = 1 + \tan^2 x$$
$$\cos ec^2 x = 1 + \cot^2 x$$

Example 1 Use $\sin^2 x + \cos^2 x = 1$ to prove $\sec^2 x = 1 + \tan^2 x$ and $\cos ec^2 x = 1 + \cot^2 x$.

$$1 = \cos^2 x + \sin^2 x, \quad \frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x},$$
$$\frac{1}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}, \quad \therefore \sec^2 x = 1 + \tan^2 x.$$

$$1 = \sin^2 x + \cos^2 x, \quad \frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x},$$
$$\frac{1}{\sin^2 x} = \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}, \quad \therefore \cos ec^2 x = 1 + \cot^2 x.$$

Example 2 Prove $\frac{1}{1 - \sin x} = \sec x(\sec x + \tan x)$.

$$RHS = \frac{1}{\cos x}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) = \frac{1}{\cos x}\left(\frac{1 + \sin x}{\cos x}\right)$$
$$= \frac{1 + \sin x}{\cos^2 x} = \frac{1 + \sin x}{1 - \sin^2 x} = \frac{1 + \sin x}{(1 - \sin x)(1 + \sin x)}$$
$$= \frac{1}{1 - \sin x} = LHS.$$

Compound angle formulas

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Double angle formulas

$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\cos 2A = 2\cos^2 A - 1$$
$$\cos 2A = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Example 1 Prove $\tan x + \tan y = \dfrac{\sin(x+y)}{\cos x \cos y}$.

$$LHS = \frac{\sin x}{\cos x} + \frac{\sin y}{\cos y} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}$$
$$= \frac{\sin(x+y)}{\cos x \cos y} = RHS$$

Example 2 Use the compound angle formulas for $\sin(A \pm B)$ and $\cos(A \pm B)$ to prove that for $\tan(A \pm B)$.

$$\tan(A \pm B) = \frac{\sin(A \pm B)}{\cos(A \pm B)} = \frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B \mp \sin A \sin B}$$
$$= \frac{\dfrac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B \mp \sin A \sin B}{\cos A \cos B}} = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} \pm \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} \mp \dfrac{\sin A \sin B}{\cos A \cos B}}$$
$$= \frac{\dfrac{\sin A}{\cos A} \pm \dfrac{\sin B}{\cos B}}{1 \mp \dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}} = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$$

Example 3
Prove $(\sin x + \sin y)(\sin x - \sin y) = \sin(x+y)\sin(x-y)$.

$$RHS = (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y)$$
$$= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$$
$$= \sin^2 x(1 - \sin^2 y) - (1 - \sin^2 x)\sin^2 y$$
$$= \sin^2 x - \sin^2 y$$
$$= (\sin x + \sin y)(\sin x - \sin y) = LHS$$

Example 4 If $\sin x = \dfrac{1}{3}$ and $\sec y = \dfrac{5}{4}$, where $x, y \in \left[0, \dfrac{\pi}{2}\right]$, evaluate $\sin(x-y)$.

Note: x and y are in the first quadrant, $\therefore \cos x$ and $\sin y$ have positive values.

$\sin x = \dfrac{1}{3}$, $\therefore \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \left(\dfrac{1}{3}\right)^2} = \sqrt{\dfrac{8}{9}} = \dfrac{2\sqrt{2}}{3}$.

$\sec y = \dfrac{5}{4}$, $\therefore \cos y = \dfrac{4}{5}$, and

$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - \left(\dfrac{4}{5}\right)^2} = \dfrac{3}{5}$.

$$\therefore \sin(x-y) = \sin x \cos y - \cos x \sin y = \frac{1}{3} \times \frac{4}{5} - \frac{2\sqrt{2}}{3} \times \frac{3}{5}$$
$$= \frac{2(2 - 3\sqrt{2})}{15}.$$

Example 5 Find the exact value of $\cos\dfrac{\pi}{12}$.

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$
$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{1 + \sqrt{3}}{2\sqrt{2}}$$
$$= \frac{\sqrt{2}(1 + \sqrt{3})}{4}.$$

Example 6 Find the exact values of $\sin\dfrac{\pi}{8}$ and $\tan\dfrac{\pi}{8}$.

For $\sin\dfrac{\pi}{8}$, consider $\cos\dfrac{\pi}{4} = \cos 2\left(\dfrac{\pi}{8}\right)$,

$\therefore \dfrac{1}{\sqrt{2}} = 1 - 2\sin^2\left(\dfrac{\pi}{8}\right)$, $\therefore \sin^2\left(\dfrac{\pi}{8}\right) = \dfrac{2 - \sqrt{2}}{4}$.

Hence $\sin\dfrac{\pi}{8} = \dfrac{\sqrt{2 - \sqrt{2}}}{2}$, since $0 < \dfrac{\pi}{8} < \dfrac{\pi}{2}$.

For $\tan\dfrac{\pi}{8}$, consider $\tan\dfrac{\pi}{4} = \tan 2\left(\dfrac{\pi}{8}\right)$,

$\therefore 1 = \dfrac{2\tan\dfrac{\pi}{8}}{1 - \tan^2\left(\dfrac{\pi}{8}\right)}$, $\therefore \tan^2\left(\dfrac{\pi}{8}\right) + 2\tan\dfrac{\pi}{8} - 1 = 0$.

Use the quadratic formula to find

$\tan\dfrac{\pi}{8} = \dfrac{-2 + \sqrt{4 + 4}}{2} = \sqrt{2} - 1$.

Example 7 Prove $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

$$LHS = \cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$$
$$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$
$$= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$$
$$= (2\cos^2\theta - 1)\cos\theta - 2(1 - \cos^2\theta)\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$
$$= 4\cos^3\theta - 3\cos\theta = RHS$$

Example 8 Solve $2 + \cos 2\beta = 3\cos\beta$ where $\beta \in [0, 2\pi]$.

$2 + \cos 2\beta = 3\cos\beta$, $2 + 2\cos^2\beta - 1 = 3\cos\beta$,

$2\cos^2\beta - 3\cos\beta + 1 = 0$,

$(2\cos\beta - 1)(\cos\beta - 1) = 0$,

$\therefore \cos\beta = \dfrac{1}{2}$, $\beta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$,

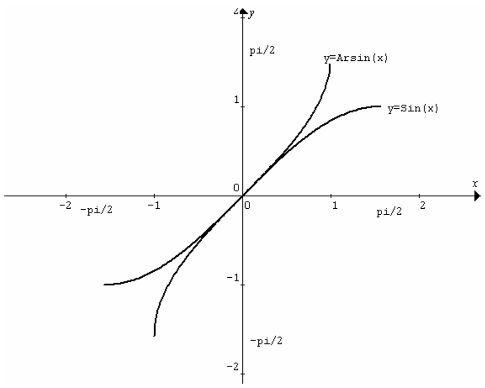
or $\cos\beta = 1$, $\beta = 0, 2\pi$.

The solution set is $\left\{0, \dfrac{\pi}{3}, \dfrac{5\pi}{3}, 2\pi\right\}$.

Restricted trigonometric functions

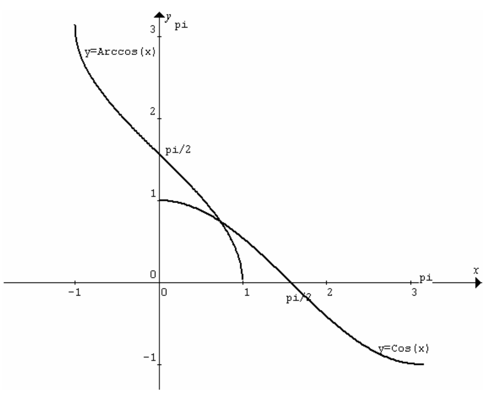
Sin x

The function $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to R, f(x)=\sin x$ is represented simply as *Sin x*. Its range is [-1, 1]. It is a one-to-one function and therefore its inverse is also a function. The inverse function is denoted as $Sin^{-1}x$. The domain of $Sin^{-1}x$ is [-1, 1] and the range is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.



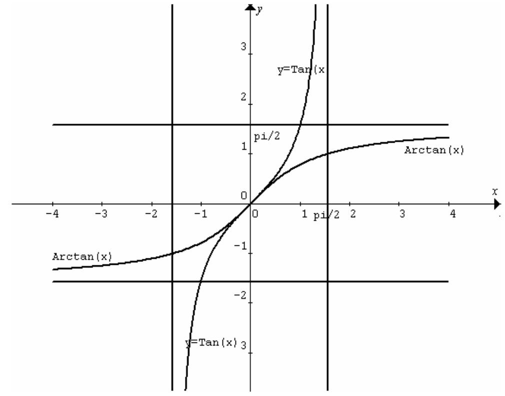
Cos x

The function $f:[0,\pi]\to R, f(x)=\cos x$ is represented by *Cos x*. Its range is [-1,1].
It is also a one-to-one function and therefore its inverse is a function. The inverse function is denoted as $Cos^{-1}x$ with domain [-1, 1] and range $[0,\pi]$.



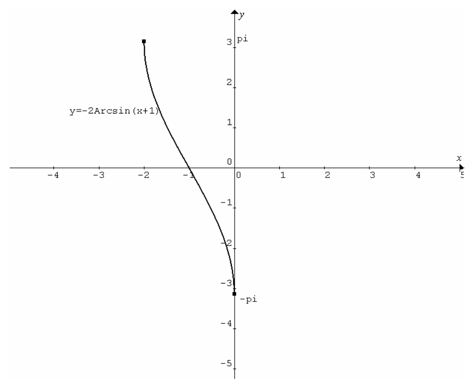
Tan x

The function $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R, f(x)=\tan x$ is represented by $Tan\,x$. Its range is *R*. It is a one-to-one function and its inverse is also a function. The inverse function is denoted as $Tan^{-1}x$ with domain *R* and range $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.



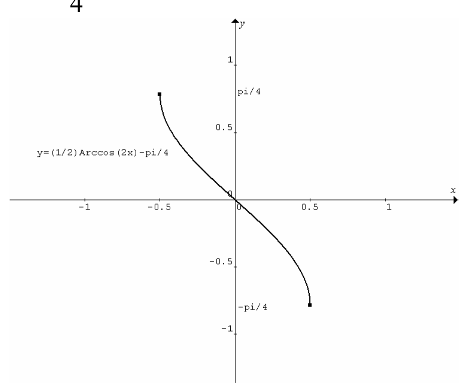
Transformations of inverse trigonometric functions

Example 1 Sketch $y=-2Sin^{-1}(x+1)$. State the domain and range.
Start with the graph of $y=Sin^{-1}x$. Reflect in the *x*-axis, dilate vertically by a factor of 2, then translate to the left by a unit.



Domain: $[-2,0]$; range: $[-\pi,\pi]$.

Example 2 Sketch $y=\frac{1}{2}Cos^{-1}(2x)-\frac{\pi}{4}$. State the domain and range.
Start with the graph of $y=Cos^{-1}x$, dilate vertically by a factor of $\frac{1}{2}$ and horizontally by the same factor, then translate downwards by $\frac{\pi}{4}$.

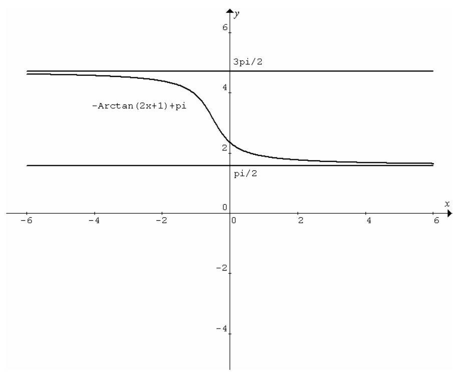


Domain: $\left[-\frac{1}{2},\frac{1}{2}\right]$; range: $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$.

Example 3 Sketch $y = -Tan^{-1}(2x+1)+\pi$. State the domain and range.

Rewrite $y = -Tan^{-1}(2x+1)+\pi$ as $y = -Tan^{-1}2\left(x+\frac{1}{2}\right)+\pi$.

Start with the graph of $y = Tan^{-1}x$, reflect in the x-axis, dilate horizontally by a factor of $\frac{1}{2}$, then translate to the left by $\frac{1}{2}$ and upwards by π .

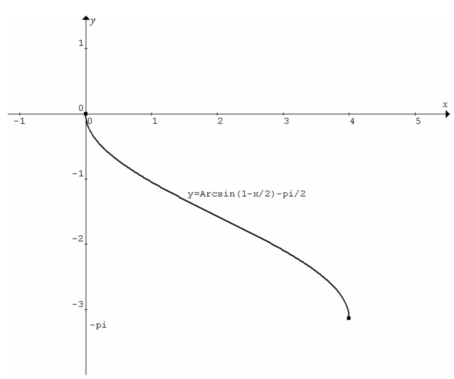


Domain: R; range: $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$.

Example 4 Sketch $y = Sin^{-1}\left(1-\frac{x}{2}\right)-\frac{\pi}{2}$. State the domain and range.

Rewrite $y = Sin^{-1}\left(1-\frac{x}{2}\right)-\frac{\pi}{2}$ as $y = Sin^{-1}\left(-\frac{1}{2}(x-2)\right)-\frac{\pi}{2}$.

Start with the graph of $y = Sin^{-1}x$, reflect in the y-axis, dilate horizontally by a factor of 2, then translate to the right by 2 units, and downwards by $\frac{\pi}{2}$.



Domain: $[0,4]$; range: $[-\pi,0]$.