# Further Calculus Part I

**Review of basic properties and rules of differentiation**

***Properties of differentiation*:**

1) $\frac{d}{dx}(kf(x))=k\frac{d}{dx}(f(x))$

2) $\frac{d}{dx}(f(x)\pm g(x))=\frac{d}{dx}(f(x))\pm\frac{d}{dx}(g(x))$

Example 1 Find the derivative of $\frac{\sqrt{3}}{2\pi}\left(2e^{\pi x}-\sqrt{3}x^{\pi}\right)$.

$\frac{d}{dx}\frac{\sqrt{3}}{2\pi}\left(2e^{\pi x}-\sqrt{3}x^{\pi}\right)=\frac{\sqrt{3}}{2\pi}\frac{d}{dx}\left(2e^{\pi x}-\sqrt{3}x^{\pi}\right)$

$=\frac{\sqrt{3}}{2\pi}\left(2\frac{d}{dx}e^{\pi x}-\sqrt{3}\frac{d}{dx}x^{\pi}\right)$

$=\frac{\sqrt{3}}{2\pi}\left(2\pi e^{\pi x}-\sqrt{3}\pi x^{\pi-1}\right)$

$=\sqrt{3}e^{\pi x}-\frac{3}{2}x^{\pi-1}$

Example 2 Given $f'(x)=\cos^2 x$ and $y=2\left(f(x)-\frac{x}{2}\right)$, evaluate $\frac{dy}{dx}$ when $x=\frac{\pi}{2}$.

$y=2\left(f(x)-\frac{x}{2}\right)$, $\frac{dy}{dx}=2\left(f'(x)-\frac{1}{2}\right)=2f'(x)-1$

$=2\cos^2 x-1=\cos 2x=\cos\pi=-1$.

***Rules of differentiation*:**

**The product rule** is used to differentiate functions that are the products of two other functions.

If $y=u(x)v(x)$, then $\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$.

Alternative notations: $y'=vu'+uv'$.

Example 1 Differentiate $2e^{\frac{x}{2}}\sin\left(2x-\frac{\pi}{2}\right)$.

$\frac{d}{dx}2e^{\frac{x}{2}}\sin\left(2x-\frac{\pi}{2}\right)$

$=2e^{\frac{x}{2}}\frac{d}{dx}\sin\left(2x-\frac{\pi}{2}\right)+\sin\left(2x-\frac{\pi}{2}\right)\frac{d}{dx}2e^{\frac{x}{2}}$

$=4e^{\frac{x}{2}}\cos\left(2x-\frac{\pi}{2}\right)+e^{\frac{x}{2}}\sin\left(2x-\frac{\pi}{2}\right)$

$=e^{\frac{x}{2}}\left[4\cos\left(2x-\frac{\pi}{2}\right)+\sin\left(2x-\frac{\pi}{2}\right)\right]$.

Example 2 Find $f'(x)$, given $f(x)=\sqrt{2x+1}\left(x^2+x-1\right)$. Hence evaluate $f'(0)$.

$f(x)=\sqrt{2x+1}\left(x^2+x-1\right)$,

$f'(x)=\sqrt{2x+1}\frac{d}{dx}\left(x^2+x-1\right)+\left(x^2+x-1\right)\frac{d}{dx}\sqrt{2x+1}$

$=\sqrt{2x+1}(2x+1)+\left(x^2+x-1\right)\frac{1}{\sqrt{2x+1}}$

$=\sqrt{2x+1}(2x+1)+\left(x^2+x-1\right)\frac{\sqrt{2x+1}}{2x+1}$

$=\sqrt{2x+1}\left(\frac{(2x+1)^2+\left(x^2+x-1\right)}{2x+1}\right)$

$=\sqrt{2x+1}\left(\frac{5x^2+5x}{2x+1}\right)=\frac{5x(x+1)\sqrt{2x+1}}{2x+1}$.

Hence $f'(0)=0$.

**The quotient rule** is used to differentiate expressions of the form $\frac{u(x)}{v(x)}$.

If $y=\frac{u(x)}{v(x)}$, then $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$.

Alternative notations: $y'=\frac{vu'-uv'}{v^2}$.

Example 1 Differentiate $\frac{2\log_e bx}{x^2}$.

$\frac{d}{dx}\left(\frac{2\log_e bx}{x^2}\right)=\frac{x^2\frac{d}{dx}(2\log_e bx)-(2\log_e bx)\frac{d}{dx}x^2}{\left(x^2\right)^2}$

$=\frac{x^2\left(\frac{2}{x}\right)-(2\log_e bx)(2x)}{x^4}$

$=\frac{2x(1-2\log_e bx)}{x^4}=\frac{2(1-2\log_e bx)}{x^3}$.

Example 2 Find an equation of the tangent line to the curve $y=\frac{x}{1+x^2}$ at $x=2$.

Gradient function: $\frac{dy}{dx}=\frac{\left(1+x^2\right)(1)-x(2x)}{\left(1+x^2\right)^2}=\frac{1-x^2}{\left(1+x^2\right)^2}$.

At $x=2$, $m_T=\frac{1-2^2}{\left(1+2^2\right)^2}=-0.12$ and $y=0.4$.

Equation of the tangent: $y-y_1=m_T(x-x_1)$,

$y-0.4=-0.12(x-2)$,

$y=-0.12x+0.28$.

**The chain rule** is used to find the derivatives of composite functions. If $y = f(u(x))$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Alternative notations: $y' = f'(u)u'$.

Example 1 Differentiate $\sqrt{\sin\left(x - \frac{\pi}{3}\right)}$.

Let $y = \sqrt{u}$, where $u = \sin\left(x - \frac{\pi}{3}\right)$.

$\frac{dy}{du} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{\sin\left(x - \frac{\pi}{3}\right)}}$ and $\frac{du}{dx} = \cos\left(x - \frac{\pi}{3}\right)$.

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{\cos\left(x - \frac{\pi}{3}\right)}{2\sqrt{\sin\left(x - \frac{\pi}{3}\right)}}$.

Example 2 Find the derivative of $\sin\sqrt{x - \frac{\pi}{3}}$.

Let $y = \sin u$, where $u = \sqrt{x - \frac{\pi}{3}}$.

$\frac{dy}{du} = \cos u = \cos\sqrt{x - \frac{\pi}{3}}$ and $\frac{du}{dx} = \frac{1}{2\sqrt{x - \frac{\pi}{3}}}$.

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{\cos\sqrt{x - \frac{\pi}{3}}}{2\sqrt{x - \frac{\pi}{3}}}$.

**Derivatives of inverse trigonometric (circular) functions**

***Inverse sine functions*:**

| $f(x)$ | $f'(x)$ |
|---|---|
| $Sin^{-1}x$ | $\frac{1}{\sqrt{1 - x^2}}$ |
| $Sin^{-1}(ax)$ | $\frac{a}{\sqrt{1 - (ax)^2}}$ |
| $Sin^{-1}\left(\frac{x}{a}\right), a > 0$ | $\frac{1}{\sqrt{a^2 - x^2}}$ |
| $Sin^{-1}\left(\frac{x}{a}\right), a < 0$ | $-\frac{1}{\sqrt{a^2 - x^2}}$ |

Example 1 Differentiate $\frac{1}{2}Sin^{-1}(-2x)$.

Let $f(x) = \frac{1}{2}Sin^{-1}(-2x)$

$f'(x) = \frac{1}{2}\frac{-2}{\sqrt{1 - (-2x)^2}} = -\frac{1}{\sqrt{1 - 4x^2}}$.

Example 2 Find $\frac{d}{dx}\left(2Sin^{-1}\frac{x}{3} + \frac{\pi}{6}\right)$.

$\frac{d}{dx}\left(2Sin^{-1}\frac{x}{3} + \frac{\pi}{6}\right) = 2\frac{d}{dx}Sin^{-1}\frac{x}{3} = \frac{2}{\sqrt{9 - x^2}}$.

Example 3 Find the derivative of $Sin^{-1}\left(\frac{2}{x}\right)$.

Let $y = Sin^{-1}u$, where $u = \frac{2}{x}$.

$\frac{dy}{du} = \frac{1}{\sqrt{1 - u^2}} = \frac{1}{\sqrt{1 - \left(\frac{2}{x}\right)^2}} = \frac{1}{\sqrt{\frac{1}{x^2}}\sqrt{x^2 - 4}} = \frac{|x|}{\sqrt{x^2 - 4}}$

and $\frac{du}{dx} = -\frac{2}{x^2} = -\frac{2}{|x|^2}$.

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{2}{|x|\sqrt{x^2 - 4}}$.

***Inverse cosine functions*:**

| $f(x)$ | $f'(x)$ |
|---|---|
| $Cos^{-1}x$ | $-\frac{1}{\sqrt{1 - x^2}}$ |
| $Cos^{-1}(ax)$ | $-\frac{a}{\sqrt{1 - (ax)^2}}$ |
| $Cos^{-1}\left(\frac{x}{a}\right), a > 0$ | $-\frac{1}{\sqrt{a^2 - x^2}}$ |
| $Cos^{-1}\left(\frac{x}{a}\right), a < 0$ | $\frac{1}{\sqrt{a^2 - x^2}}$ |

Example 1

Show that the derivative of $Cos^{-1}\left(\frac{ax}{b}\right)$ is $-\frac{a}{\sqrt{b^2 - (ax)^2}}$ for $a, b > 0$. Hence differentiate $Cos^{-1}\left(\frac{2x}{3}\right)$.

Let $y = Cos^{-1}(u)$, where $u = \frac{ax}{b}$.

$\frac{dy}{du} = -\frac{1}{\sqrt{1 - u^2}} = -\frac{1}{\sqrt{1 - \left(\frac{ax}{b}\right)^2}} = -\frac{b}{\sqrt{b^2 - (ax)^2}}$ and

$\frac{du}{dx} = \frac{a}{b}$.

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{a}{\sqrt{b^2 - (ax)^2}}$.

Hence $\frac{d}{dx}Cos^{-1}\left(\frac{2x}{3}\right) = -\frac{2}{\sqrt{9 - 4x^2}}$.

***Inverse tangent functions*:**

| $f(x)$ | $f'(x)$ |
|---|---|
| $Tan^{-1}x$ | $\dfrac{1}{1+x^2}$ |
| $Tan^{-1}(ax)$ | $\dfrac{a}{1+(ax)^2}$ |
| $Tan^{-1}\left(\dfrac{x}{a}\right)$ | $\dfrac{a}{a^2+x^2}$ |

Example 1  Show that the derivative of $Tan^{-1}\left(\dfrac{ax}{b}\right)$ is $\dfrac{ab}{b^2+(ax)^2}$. Hence differentiate $Tan^{-1}\left(\dfrac{-3x}{2}\right)$.

Let $y=Tan^{-1}(u)$, where $u=\dfrac{ax}{b}$.

$$\frac{dy}{du}=\frac{1}{1+u^2}=\frac{1}{1+\left(\frac{ax}{b}\right)^2}=\frac{b^2}{b^2+(ax)^2} \text{ and } \frac{du}{dx}=\frac{a}{b}.$$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{ab}{b^2+(ax)^2}.$$

Hence $\dfrac{d}{dx}Tan^{-1}\left(\dfrac{-3x}{2}\right)=-\dfrac{6}{4+9x^2}$.

**Second derivatives**

The derivative of the derivative of a function is called the second derivative of the function.

If $y=f(x)$, the second derivative is denoted by

$$y''=f''(x)=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}.$$

Example 1  Find $\dfrac{d^2y}{dx^2}$, given $y=\log_e b(x+1)$.

$$y=\log_e b(x+1),\ \frac{dy}{dx}=\frac{1}{x+1},\ \frac{d^2y}{dx^2}=-\frac{1}{(x+1)^2}.$$

Example 2  Show that $\dfrac{d^2y}{dx^2}=4y$ for $y=3e^{-2x}$.

$$y=3e^{-2x},\ \frac{dy}{dx}=-6e^{-2x},$$

$$\frac{d^2y}{dx^2}=12e^{-2x}=4\times 3e^{-2x}=4y.$$

Example 3  The position of an object moving along a straight path is given by $x=5-10\cos^2 t$ for $t\ge 0$. Find the position, velocity and acceleration of the object at $t=\dfrac{\pi}{2}$.

$$x=5-10\cos^2 t=-5(2\cos^2 t-1)=-5\cos 2t,$$

$$v=\frac{dx}{dt}=10\sin 2t,\ a=\frac{dv}{dt}=\frac{d^2x}{dt^2}=20\cos 2t.$$

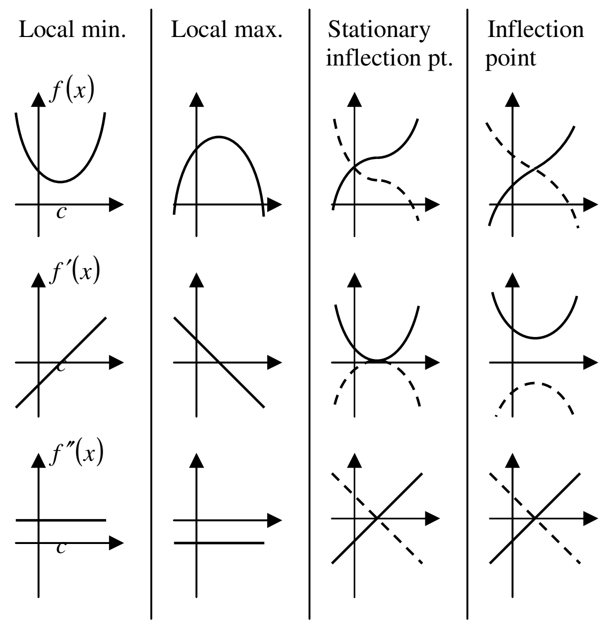
At $t=\dfrac{\pi}{2}$, $x=-5\cos\pi=5$,

$v=10\sin\pi=0$, $a=20\cos\pi=-20$.

**Application of second derivative to the analysis of graphs of functions**

Given a **continuous** and 'smooth' function $f(x)$,

| $f'(c)$ | $f''(c)$ | at $x=c$ |
|---|---|---|
| $=0$ | $>0$ | local minimum |
| $=0$ | $<0$ | local maximum |
| $=0$ | $=0$ | stationary inflection point |
| $\ne 0$ | $=0$ | inflection point |




Example 1  Find the stationary points and inflection points of the function $e^x\sin x$, $0\le x\le 2\pi$. Sketch the graph.

Let $y=e^x\sin x$, $\dfrac{dy}{dx}=e^x\cos x+e^x\sin x=e^x(\cos x+\sin x)$,

$$\frac{d^2y}{dx^2}=e^x(-\sin x+\cos x)+e^x(\cos x+\sin x)=2e^x\cos x.$$

Inflection points: $\dfrac{d^2y}{dx^2}=0$, $\therefore 2e^x\cos x=0$, $\therefore \cos x=0$,

$x=\dfrac{\pi}{2},\dfrac{3\pi}{2}$.

$\therefore y=e^{\frac{\pi}{2}},-e^{\frac{3\pi}{2}}$.  $\left(\dfrac{\pi}{2},e^{\frac{\pi}{2}}\right)$, $\left(\dfrac{3\pi}{2},-e^{\frac{3\pi}{2}}\right)$.

Stationary points: $\frac{dy}{dx}=0$ , $\therefore e^x(\cos x+\sin x)=0$ ,

$\therefore \cos x+\sin x=0$ , $1+\tan x=0$ , $\tan x=-1$ ,

$\therefore x=\frac{3\pi}{4},\frac{7\pi}{4}$ .

$\therefore y=e^{\frac{3\pi}{4}}\sin\frac{3\pi}{4}=\frac{\sqrt{2}e^{\frac{3\pi}{4}}}{2}$ or $e^{\frac{7\pi}{4}}\sin\frac{7\pi}{4}=-\frac{\sqrt{2}e^{\frac{7\pi}{4}}}{2}$ .

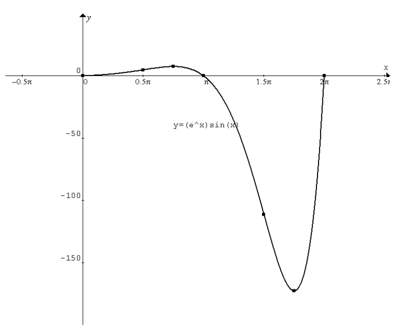
$\left(\frac{3\pi}{4},\frac{\sqrt{2}e^{\frac{3\pi}{4}}}{2}\right)$, $\left(\frac{7\pi}{4},-\frac{\sqrt{2}e^{\frac{7\pi}{4}}}{2}\right)$.

At $x=\frac{3\pi}{4}$ , $\frac{d^2y}{dx^2}=2e^x\cos x=2e^{\frac{3\pi}{4}}\cos\frac{3\pi}{4}<0$ ,

$\therefore \left(\frac{3\pi}{4},\frac{\sqrt{2}e^{\frac{3\pi}{4}}}{2}\right)$ is a local maximum.

At $x=\frac{7\pi}{4}$ , $\frac{d^2y}{dx^2}=2e^x\cos x=2e^{\frac{7\pi}{4}}\cos\frac{7\pi}{4}>0$ ,

$\therefore \left(\frac{7\pi}{4},-\frac{\sqrt{2}e^{\frac{7\pi}{4}}}{2}\right)$ is a local minimum.




Example 2 Find and determine the nature of the stationary points and inflection points of the function with equation $y=\frac{x}{x^2+1}$ . Does it have an asymptote? Sketch its graph.

$y=\frac{x}{x^2+1}$ , $y'=\frac{vu'-uv'}{v^2}=\frac{(x^2+1)-x(2x)}{(x^2+1)^2}=\frac{1-x^2}{(x^2+1)^2}$ ,

$y''=\frac{(x^2+1)^2(-2x)-(1-x^2)(2(x^2+1)2x)}{(x^2+1)^4}$

$=\frac{2x(x^2+1)(x^2-3)}{(x^2+1)^4}$.

Stationary points: $y'=0$ , $\frac{1-x^2}{(x^2+1)^2}=0$ , $\therefore 1-x^2=0$ ,

$\therefore x=-1,1$ , $\therefore y=-\frac{1}{2},\frac{1}{2}$ . $\left(-1,-\frac{1}{2}\right)$, $\left(1,\frac{1}{2}\right)$.

At $x=-1$ , $y''>0$ , $\therefore \left(-1,-\frac{1}{2}\right)$ is a local minimum.

At $x=1$ , $y''<0$ , $\therefore \left(1,\frac{1}{2}\right)$ is a local maximum.

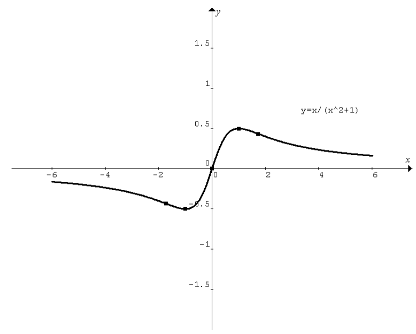
Inflection points: $y''=0$ ,

$\frac{2x(x^2+1)(x^2-3)}{(x^2+1)^4}=0$ ,

$\therefore 2x(x^2+1)(x^2-3)=0$ ,

$\therefore x=0,\pm\sqrt{3}$ , $\therefore y=0,\pm\frac{\sqrt{3}}{4}$ .

$(0,0)$ , $\left(-\sqrt{3},-\frac{\sqrt{3}}{4}\right)$, $\left(\sqrt{3},\frac{\sqrt{3}}{4}\right)$.




Horizontal asymptote: $y=0$ .

**Implicit differentiation**

This topic is an application of the chain rule. The derivatives of relations like $x^2+y^2=9$ and $3xy^2=x+y$ can be obtained by implicit differentiation. This method eliminates the tedious task of rewriting a relation to make *y* the subject.

Example 1 Given $x^2+y^2=9$ , use implicit differentiation to find $\frac{dy}{dx}$ in terms of *x*.

Differentiate both sides of $x^2+y^2=9$ with respect to *x*, use the chain rule if necessary. Note: *y* is a function of *x*.

$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(9)$,

$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=0$ ,

$2x+\frac{d(y^2)}{dy}\times\frac{dy}{dx}=0$ ,

$\therefore 2x+2y\frac{dy}{dx}=0$ , $\frac{dy}{dx}=-\frac{x}{y}$ .

Since $y=\pm\sqrt{9-x^2}$ , $\therefore \frac{dy}{dx}=\pm\frac{x}{\sqrt{9-x^2}}$ .

Example 2 Given $3xy^2 = x + y$, use implicit differentiation to find $\frac{dy}{dx}$ in terms of $x$ and $y$. Hence evaluate $\frac{dy}{dx}$ at $x = 2$.

Differentiate both sides of $3xy^2 = x + y$ with respect to $x$,

$\frac{d}{dx}\left(3xy^2\right) = \frac{d}{dx}(x + y)$,

$y^2\frac{d}{dx}(3x) + 3x\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(x) + \frac{d}{dx}(y)$,

$\therefore 3y^2 + 3x\frac{d\left(y^2\right)}{dy}\times\frac{dy}{dx} = 1 + \frac{dy}{dx}$, $\therefore 3y^2 + 6xy\frac{dy}{dx} = 1 + \frac{dy}{dx}$,

$6xy\frac{dy}{dx} - \frac{dy}{dx} = 1 - 3y^2$, $(6xy - 1)\frac{dy}{dx} = 1 - 3y^2$,

$\therefore \frac{dy}{dx} = \frac{1 - 3y^2}{6xy - 1}$.

At $x = 2$, $6y^2 = 2 + y$, $6y^2 - y - 2 = 0$,

$(2y + 1)(3y - 2) = 0$, $\therefore y = -\frac{1}{2}, \frac{2}{3}$.

Hence $\frac{dy}{dx} = \frac{1}{28}$, $-\frac{1}{21}$.

**Related rates**

In a **related rates** problem, the rate of change of one quantity is expressed in terms of the rate of change of another quantity. This requires one to find a relationship (i.e. an equation) between the two quantities, and using the chain rule to differentiate both sides with respect to time or other variables.

Example 1 Air is pumped into a spherical balloon at a constant rate of $80$ cm$^3$s$^{-1}$. How fast does the radius increase when it is 30 cm?

The two quantities in this problem are volume $V$ cm$^3$ and radius $r$ cm. Their relationship is $V = \frac{4}{3}\pi r^3$.

Differentiate both sides with respect to time $t$:

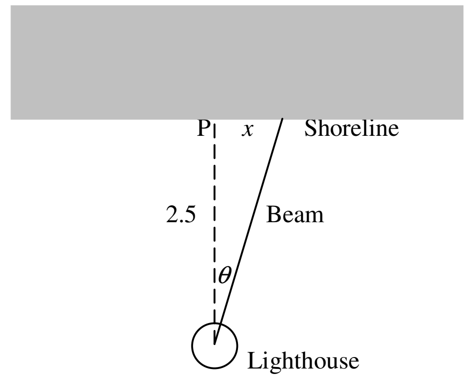
$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$,

$\therefore \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$.

Given $\frac{dV}{dt} = +80$, at $r = 30$,

$\frac{dr}{dt} = \frac{1}{4\pi(30)^2}\times 80 = \frac{1}{45\pi}$ cms$^{-1}$.

Example 2 A lighthouse is on a small island 2.5 km from the nearest point P on a straight shoreline. The light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 0.5 km from P?




Let $x$ km be the distance of the beam from P along the shoreline, and $\theta$ the angle in radians between the beam and the dotted line.
The two quantities in this problem are $x$ and $\theta$. Their relationship is $x = 2.5\tan\theta$.

Differentiate both sides with respect to time $t$ (hours):

$\frac{dx}{dt} = \frac{d}{dt}(2.5\tan\theta) = \frac{d}{d\theta}(2.5\tan\theta)\frac{d\theta}{dt} = 2.5\sec^2\theta\frac{d\theta}{dt}$.

Given $\frac{d\theta}{dt} = 4$ rev min$^{-1}$ $= 8\pi$ radians min$^{-1}$

$= 480\pi$ radians hour$^{-1}$, and at $x = 0.5$,

$\theta = \tan^{-1}\left(\frac{0.5}{2.5}\right) = 0.1974$.

$\therefore \frac{dx}{dt} = 2.5\left(\sec^2 0.1974\right)\times 480\pi = 3921$ kmh$^{-1}$.

Alternative method:

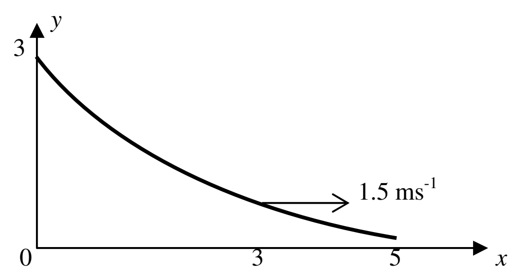
Express the relationship as $\theta = \tan^{-1}\left(\frac{x}{2.5}\right)$.

Differentiate both sides with respect to time $t$ (hours):

$\frac{d\theta}{dt} = \frac{d}{dt}\tan^{-1}\left(\frac{x}{2.5}\right) = \frac{d}{dx}\tan^{-1}\left(\frac{x}{2.5}\right)\frac{dx}{dt} = \frac{2.5}{2.5^2 + x^2}\frac{dx}{dt}$,

$\therefore \frac{dx}{dt} = \frac{2.5^2 + x^2}{2.5}\frac{d\theta}{dt} = \frac{2.5^2 + 0.5^2}{2.5}\times 480\pi = 3921$ kmh$^{-1}$.

Example 3 A playground slide has a profile given by $y = 3e^{-\frac{x}{2}}$, $x \in [0,5]$. A kid slides down the slope and has a horizontal speed of 1.5 ms$^{-1}$ at $x = 3$ m. Calculate the vertical speed at that instant.



The two quantities in this problem are $x$ and $y$. Their relationship is $y = 3e^{-\frac{x}{2}}$.

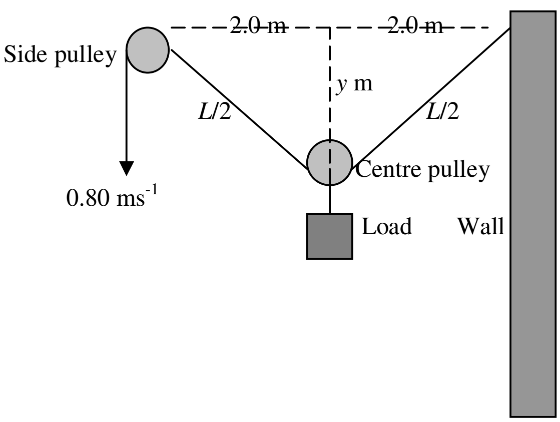
Differentiate both sides with respect to time $t$ (s):

$$\frac{dy}{dt} = \frac{d}{dt}3e^{-\frac{x}{2}} = \frac{d}{dx}3e^{-\frac{x}{2}}\frac{dx}{dt} = -\frac{3}{2}e^{-\frac{x}{2}}\frac{dx}{dt}.$$

Given $\frac{dx}{dt} = 1.5$ at $x = 3$,

$\therefore \frac{dy}{dt} = -\frac{3}{2}e^{-\frac{3}{2}} \times 1.5 = -0.50$ ms$^{-1}$, i.e. the vertical speed of the kid is 0.50 ms$^{-1}$. The negative sign indicates downward motion.

Example 4 Consider the following pulley system in lifting a load. The side pulley is 4.0 m from the wall. The rope is pulled at a constant speed of 0.80 ms$^{-1}$. How fast does the load rise when the centre pulley is 1.0 m lower than the side pulley?




Let $L$ m be the length of the rope between the side pulley and the wall when the centre pulley is $y$ m lower than the side pulley.

$$\left(\frac{L}{2}\right)^2 = 2^2 + y^2, \ \therefore y = \sqrt{\frac{L^2}{4} - 4}.$$

Differentiate both sides with respect to time $t$ (s):

$$\frac{dy}{dt} = \frac{d}{dt}\sqrt{\frac{L^2}{4} - 4} = \frac{d}{dL}\sqrt{\frac{L^2}{4} - 4}\frac{dL}{dt} = \frac{L}{4\sqrt{\frac{L^2}{4} - 4}}\frac{dL}{dt}$$

$$\therefore \frac{dy}{dt} = \frac{L}{2\sqrt{L^2 - 16}}\frac{dL}{dt}.$$

Given $\frac{dL}{dt} = -0.80$, when $y = 1.0$, $L = 2\sqrt{5}$,

$\therefore \frac{dy}{dt} = -0.89$, i.e. $y$ decreases at 0.89 ms$^{-1}$ and the load rises at 0.89 ms$^{-1}$.

**Review of some basic anti-differentiation properties**

1) $\int f'(x)dx = f(x) + c$

2) $\int kh(x)dx = k\int h(x)dx$

3) $\int (p(x) \pm q(x))dx = \int p(x)dx \pm \int q(x)dx$

Example 1 Find $\int 2\left(e^{2x} - \frac{1}{x}\right)dx$.

$$\int 2\left(e^{2x} - \frac{1}{x}\right)dx = 2\int\left(e^{2x} - \frac{1}{x}\right)dx$$
$$= 2\left(\int e^{2x}dx - \int \frac{1}{x}dx\right) = 2\left(\frac{1}{2}e^{2x} - \log_e|x| + c\right)$$
$$= e^{2x} - 2\log_e|x| + C$$

Example 2 Anti-differentiate $\frac{5x^2 - 2x + x^{-1}}{x}$.

$$\int \frac{5x^2 - 2x + x^{-1}}{x}dx = \int (5x - 2 + x^{-2})dx$$
$$= \int 5xdx - \int 2dx + \int x^{-2}dx$$
$$= \frac{5}{2}x^2 - 2x - \frac{1}{x} + C$$

Example 3 If $f'(x) = \cos 2x - \sin 2x$, find $f(x)$, given $f\left(\frac{\pi}{2}\right) = 0$.

$f'(x) = \cos 2x - \sin 2x$, $\therefore f(x) = \int (\cos 2x - \sin 2x)dx$

$= \int \cos 2xdx - \int \sin 2xdx = \frac{1}{2}\sin 2x + \frac{1}{2}\cos 2x + C$.

Given $f\left(\frac{\pi}{2}\right) = 0$, $\therefore \frac{1}{2}\sin 2\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos 2\left(\frac{\pi}{2}\right) + C = 0$,

$\therefore C = \frac{1}{2}$. Hence $f(x) = \frac{1}{2}\sin 2x + \frac{1}{2}\cos 2x + \frac{1}{2}$.

Example 4 Differentiate $\log_e(\sin(2x-1))$ and hence find an anti-derivative of $\cot(2x-1)$.

Apply the chain rule: Let $y = \log_e u$, where $u = \sin(2x-1)$,

$\frac{dy}{du} = \frac{1}{u} = \frac{1}{\sin(2x-1)}$, $\frac{du}{dx} = 2\cos(2x-1)$.

$$\therefore \frac{d}{dx}\log_e(\sin(2x-1)) = \frac{dy}{du} \times \frac{du}{dx} = \frac{2\cos(2x-1)}{\sin(2x-1)} = 2\cot(2x-1)$$

$$\therefore \cot(2x-1) = \frac{1}{2}\frac{d}{dx}\log_e(\sin(2x-1)).$$

Hence $\int \cot(2x-1)dx = \int \frac{1}{2}\frac{d}{dx}\log_e(\sin(2x-1))dx$

$$= \frac{1}{2}\int \frac{d}{dx}\log_e(\sin(2x-1))dx = \frac{1}{2}\log_e(\sin(2x-1)).$$

**Anti-differentiation techniques**

***Anti-differentiation of*** $\frac{1}{\sqrt{a^2-x^2}}$ ***and*** $\frac{a}{a^2+x^2}$

| $f(x)$ | $\int f(x)dx$ |
|---|---|
| $\frac{1}{\sqrt{a^2-x^2}}, a>0$ | $Sin^{-1}\left(\frac{x}{a}\right)+C$ |
| $\frac{1}{\sqrt{1-(ax)^2}}, a>0$ | $\frac{1}{a}Sin^{-1}(ax)+C$ |
| $\frac{-1}{\sqrt{a^2-x^2}}, a>0$ | $Cos^{-1}\left(\frac{x}{a}\right)+C$ |
| $\frac{-1}{\sqrt{1-(ax)^2}}, a>0$ | $\frac{1}{a}Cos^{-1}(ax)+C$ |
| $\frac{a}{a^2+x^2}, a>0$ | $Tan^{-1}\left(\frac{x}{a}\right)+C$ |
| $\frac{1}{1+(ax)^2}, a>0$ | $\frac{1}{a}Tan^{-1}(ax)+C$ |

Example 1 Find $\int \frac{-1}{\sqrt{2-x^2}}dx$.

$$\int \frac{-1}{\sqrt{2-x^2}}dx = \int \frac{-1}{\sqrt{\left(\sqrt{2}\right)^2-x^2}}dx = Cos^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$$

Example 2 Find an anti-derivative of $\frac{3}{5+x^2}$.

$$\int \frac{3}{5+x^2}dx = \int \frac{\frac{3}{\sqrt{5}}\times\sqrt{5}}{\left(\sqrt{5}\right)^2+x^2}dx = \frac{3}{\sqrt{5}}\int \frac{\sqrt{5}}{\left(\sqrt{5}\right)^2+x^2}dx$$

$$= \frac{3}{\sqrt{5}}Tan^{-1}\left(\frac{x}{\sqrt{5}}\right) \text{ or } \frac{3\sqrt{5}}{5}Tan^{-1}\left(\frac{x}{\sqrt{5}}\right).$$

Example 3 Anti-differentiate $\frac{1}{\sqrt{1-4x^2}}$.

$$\int \frac{1}{\sqrt{1-4x^2}}dx = \int \frac{1}{\sqrt{4\left(\frac{1}{4}-x^2\right)}}dx = \int \frac{1}{2\sqrt{\left(\frac{1}{2}\right)^2-x^2}}dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-x^2}}dx = \frac{1}{2}Sin^{-1}\left(\frac{x}{\frac{1}{2}}\right)+C = \frac{1}{2}Sin^{-1}(2x)+C.$$

Alternative method:

$$\int \frac{1}{\sqrt{1-4x^2}}dx = \int \frac{1}{\sqrt{1-(2x)^2}}dx = \frac{1}{2}Sin^{-1}(2x)+C$$

***Anti-differentiation of expressions of the form*** $f(g(x))g'(x)$ ***using the substitution*** $u=g(x)$

Note: $\frac{du}{dx}dx = du$

Example 1 Find $\int 2x\sqrt{1-3x^2}\,dx$.

Let $u = 1-3x^2$, $\frac{du}{dx} = -6x$, $\therefore x = -\frac{1}{6}\frac{du}{dx}$.

$$\therefore \int 2x\sqrt{1-3x^2}\,dx = \int 2\left(-\frac{1}{6}\right)\sqrt{u}\frac{du}{dx}dx$$

$$= -\frac{1}{3}\int u^{\frac{1}{2}}du = -\frac{1}{3}\times\frac{2}{3}u^{\frac{3}{2}}+C = -\frac{2}{9}\left(1-3x^2\right)^{\frac{3}{2}}+C.$$

Example 2 Find an anti-derivative of $\frac{(\log_e x)^2}{x}$.

Let $u = \log_e x$, $\frac{du}{dx} = \frac{1}{x}$.

$$\int \frac{(\log_e x)^2}{x}dx = \int u^2\frac{du}{dx}dx = \int u^2 du$$

$$= \frac{1}{3}u^3 + C = \frac{1}{3}(\log_e x)^3 + C.$$

Example 3 Find $\int \sec^2 x\tan^2 x\,dx$.

Let $u = \tan x$, $\frac{du}{dx} = \sec^2 x$.

$$\int \sec^2 x\tan^2 x\,dx = \int u^2\frac{du}{dx}dx = \int u^2 du$$

$$= \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C.$$

Example 4 Find $\int \frac{1}{\sqrt{1-x^2}\,Sin^{-1}x}dx$.

Let $u = Sin^{-1}x$, $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$.

$$\int \frac{1}{\sqrt{1-x^2}\,Sin^{-1}x}dx = \int \frac{1}{u}\frac{du}{dx}dx = \int \frac{1}{u}du$$

$$= \log_e|u| + C = \log_e\left|Sin^{-1}x\right| + C$$

Example 5 Anti-differentiate $\cos^2 x\sin^3 x$.

$$\int \cos^2 x\sin^3 x\,dx = \int \cos^2 x\sin^2 x\sin x\,dx$$

$$= \int \cos^2 x\left(1-\cos^2 x\right)\sin x\,dx$$

$$= \int \left(\cos^2 x - \cos^4 x\right)\sin x\,dx.$$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$, $\therefore \sin x = -\frac{du}{dx}$.

$$\therefore \int \left(\cos^2 x - \cos^4 x\right)\sin x\,dx = \int \left(u^2 - u^4\right)\left(-\frac{du}{dx}\right)dx$$

$$= \int \left(u^4 - u^2\right)du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.$$

***Anti-differentiation by linear substitution***

This technique is particularly useful for expressions of the form $f(x)g(ax+b)$, where $f(x)$ is a polynomial function and $g$ is a rational power function of $ax+b$.
Using the substitution $u=ax+b$, $f(x)g(ax+b)$ becomes $f\left(\frac{u-b}{a}\right)g(u)$.

Example 1 Find $\int x\sqrt{1-x}dx$.

Let $u=1-x$, then $x=1-u$, $\frac{du}{dx}=-1$ or $1=-\frac{du}{dx}$.

$$\int x\sqrt{1-x}dx=\int(1-u)\sqrt{u}\left(-\frac{du}{dx}\right)dx=\int-(1-u)u^{\frac{1}{2}}du$$
$$=\int\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du=\frac{2}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}+C$$
$$=\frac{2}{5}(1-x)^{\frac{5}{2}}-\frac{2}{3}(1-x)^{\frac{3}{2}}+C.$$

Example 2 Find $\int\frac{x^2+1}{\sqrt{x+1}}dx$.

Let $u=x+1$, then $x=u-1$, $\frac{du}{dx}=1$.

$$\int\frac{x^2+1}{\sqrt{x+1}}dx=\int\frac{(u-1)^2+1}{u^{\frac{1}{2}}}\frac{du}{dx}dx=\int\left(u^{\frac{3}{2}}-2u^{\frac{1}{2}}+2u^{-\frac{1}{2}}\right)du$$
$$=\frac{2}{5}u^{\frac{5}{2}}-\frac{4}{3}u^{\frac{3}{2}}+4u^{\frac{1}{2}}+C$$
$$=\frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{4}{3}(x+1)^{\frac{3}{2}}+4(x+1)^{\frac{1}{2}}+C.$$

Example 3 Find an anti-derivative of $(2x+1)(2x-3)^{\frac{2}{3}}$.

Let $u=2x-3$, then $x=\frac{u+3}{2}$, $\frac{du}{dx}=2$ or $\frac{1}{2}\frac{du}{dx}=1$.

$$\int(2x+1)(2x-3)^{\frac{2}{3}}dx=\int(u+4)u^{\frac{2}{3}}\left(\frac{1}{2}\frac{du}{dx}\right)dx$$
$$=\frac{1}{2}\int\left(u^{\frac{5}{3}}+4u^{\frac{2}{3}}\right)du=\frac{1}{2}\left(\frac{3}{8}u^{\frac{8}{3}}+\frac{12}{5}u^{\frac{5}{3}}\right)$$
$$=\frac{3}{16}(2x-3)^{\frac{8}{3}}+\frac{6}{5}(2x-3)^{\frac{5}{3}}.$$

***Anti-differentiation using the trigonometric identities***
$\sin^2(ax)=\frac{1}{2}(1-\cos(2ax))$ ***and*** $\cos^2(ax)=\frac{1}{2}(1+\cos(2ax))$

Example 1 Find $\int\cos^2\left(\frac{x}{2}\right)dx$.

$$\int\cos^2\left(\frac{x}{2}\right)dx=\int\frac{1}{2}(1+\cos x)dx=\frac{1}{2}\int(1+\cos x)dx$$
$$=\frac{1}{2}(x+\sin x+c)=\frac{1}{2}x+\frac{1}{2}\sin x+C.$$

Example 2 Anti-differentiate $2\sin^2(3x)-5x$.

$$\int\left(2\sin^2(3x)-5x\right)dx=\int(1-\cos2(3x)-5x)dx$$
$$=x-\frac{1}{6}\sin(6x)-\frac{5}{2}x^2+C.$$

Example 3 Find an anti-derivative of $\cos^4 x$.

$$\cos^4 x=\left(\cos^2 x\right)^2=\left(\frac{1}{2}(1+\cos2x)\right)^2$$
$$=\frac{1}{4}\left(1+2\cos2x+\cos^2 2x\right)$$
$$=\frac{1}{4}\left(1+2\cos2x+\frac{1}{2}(1+\cos4x)\right)$$
$$=\frac{1}{4}\left(\frac{3}{2}+2\cos2x+\frac{1}{2}\cos4x\right)$$
$$=\frac{1}{8}(3+4\cos2x+\cos4x)$$

$$\therefore\int\cos^4 xdx=\frac{1}{8}\int(3+4\cos2x+\cos4x)dx$$
$$=\frac{1}{8}\left(3x+2\sin2x+\frac{1}{4}\sin4x\right)$$
$$=\frac{1}{32}(12x+8\sin2x+\sin4x).$$

***Anti-differentiation using partial fractions of rational functions with quadratic denominators***

*Partial fractions*

A fraction can be changed to the sum or difference of two or more fractions, for examples,

$$\frac{5}{6}=\frac{1}{2}+\frac{2}{3},\quad\frac{3}{4}=\frac{1}{4}+\frac{1}{2},\quad\frac{11}{10}=1+\frac{1}{2}-\frac{4}{5}.$$

Rational algebraic expressions can also be changed to partial fractions if the denominator can be factorised.

Example 1 Change $\frac{x+1}{x^2-4}$ to partial fractions.

$$\frac{x+1}{x^2-4}=\frac{x+1}{(x-2)(x+2)}=\frac{A}{x-2}+\frac{B}{x+2}=\frac{A(x+2)+B(x-2)}{(x-2)(x+2)}.$$

Compare the numerators: $A(x+2)+B(x-2)=x+1$, it is true for all $x\in R$.

Let $x=2$, $4A=3$, $A=\frac{3}{4}$.

Let $x=-2$, $-4B=-1$, $B=\frac{1}{4}$.

Hence $\frac{x+1}{x^2-4}=\frac{\frac{3}{4}}{x-2}+\frac{\frac{1}{4}}{x+2}$ or $\frac{3}{4(x-2)}+\frac{1}{4(x+2)}$.

Example 2 Convert $\frac{x}{(x-3)^2}$ to partial fractions.

The rational expression has a repeated factor.
$\frac{x}{(x-3)^2}=\frac{A}{x-3}+\frac{B}{(x-3)^2}=\frac{A(x-3)+B}{(x-3)^2}$ $\therefore A(x-3)+B=x$.

Let $x=3$, $\therefore B=3$. Let $x=0$, $\therefore 3A=B=3$, $\therefore A=1$.
Hence $\frac{x}{(x-3)^2}=\frac{1}{x-3}+\frac{3}{(x-3)^2}$.

Example 3 Transform $\frac{2x^2+3}{x^2-3x+2}$ to partial fractions.

The numerator and denominator have the same degree.
$\frac{2x^2+3}{x^2-3x+2}=\frac{2(x^2-3x+2)+6x-1}{x^2-3x+2}=2+\frac{6x-1}{x^2-3x+2}$
$=2+\frac{6x-1}{(x-2)(x-1)}=2+\frac{A}{x-2}+\frac{B}{x-1}$
$=2+\frac{A(x-1)+B(x-2)}{(x-2)(x-1)}$. $\therefore A(x-1)+B(x-2)=6x-1$.

Let $x=2$, $A=11$. Let $x=1$, $B=-5$.
Hence $\frac{2x^2+3}{x^2-3x+2}=2+\frac{11}{x-2}-\frac{5}{x-1}$.

Example 4 Find $\int\frac{x+1}{x^2-4}dx$.
Refer to example 1.
$\int\frac{x+1}{x^2-4}dx=\int\left(\frac{\frac{3}{4}}{x-2}+\frac{\frac{1}{4}}{x+2}\right)dx$
$=\frac{3}{4}\log_e|x-2|+\frac{1}{4}\log_e|x+2|+C$

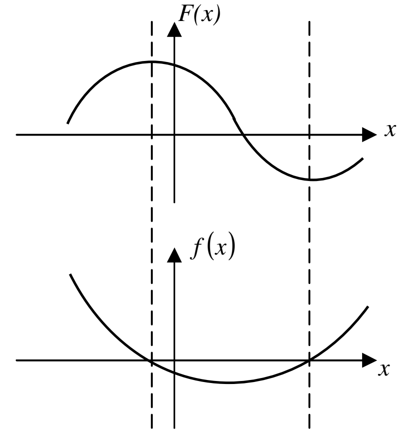
Example 5 Find $F(x)=\int f(x)dx$, given $f(x)=\frac{x}{(x-3)^2}$ and $F(4)=4$.
Refer to example 2, $f(x)=\frac{x}{(x-3)^2}=\frac{1}{x-3}+\frac{3}{(x-3)^2}$.
$F(x)=\int f(x)dx=\int\left(\frac{1}{x-3}+\frac{3}{(x-3)^2}\right)dx$
$=\log_e|x-3|-\frac{3}{x-3}+C$.
Given $F(4)=4$, $\therefore F(4)=\log_e|4-3|-\frac{3}{4-3}+C=4$,
$\therefore C=7$. Hence $F(x)=\log_e|x-3|-\frac{3}{x-3}+7$.

Example 6 Anti-differentiate $\frac{2x^2+3}{x^2-3x+2}$.
Refer to example 3.
$\int\frac{2x^2+3}{x^2-3x+2}dx=\int\left(2+\frac{11}{x-2}-\frac{5}{x-1}\right)dx$
$=2x+11\log_e|x-2|-5\log_e|x-1|+C$.

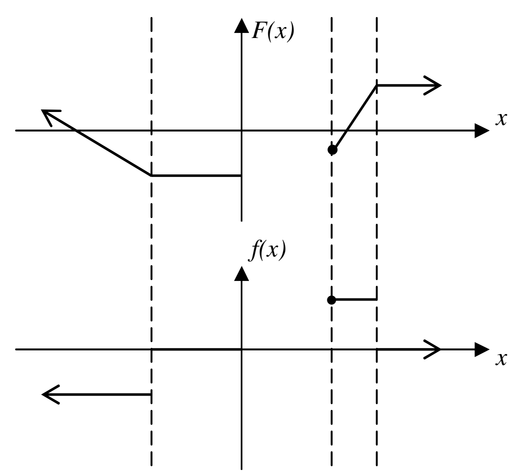
**Relationship between the graph of a function and the graphs of its anti-derivatives**

The graph of the gradient of an anti-derivative is the graph of the original function, $\because \frac{d}{dx}\int f(x)dx=f(x)$, where $f(x)$ is the original function.

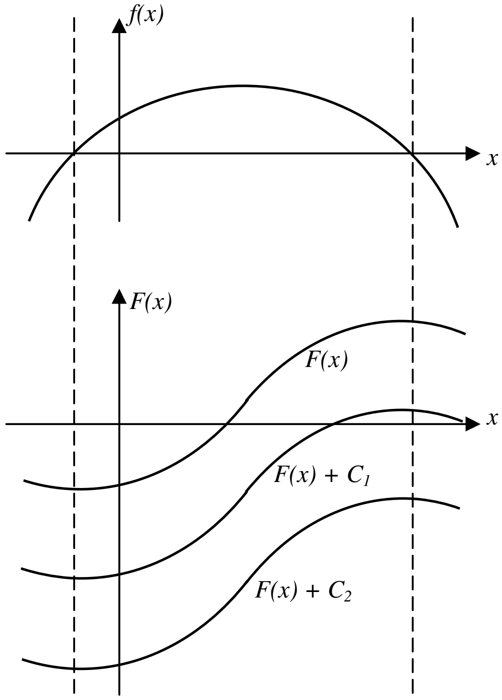
Example 1 Sketch the graph of $f(x)$ given the graph of $F(x)=\int f(x)dx$ shown below.




Example 2 Given the graph of an anti-derivative of a function as shown below, sketch the graph of the function.



Example 3 Sketch the graph of an anti-derivative of the function $f(x)$ shown below.




On the above graph sketch another two anti-derivative graphs of the given function $f(x)$.

**Evaluation of definite integrals**

In cases involving substitution and change of variable *x* to *u*, it is not necessary to convert the anti-derivative back to the original variable *x* before evaluation. It is simpler to change the values of the lower and upper limits of the definite integral to the values corresponding to the new variable *u*.

Example 1 Evaluate $\int_1^e \frac{\log_e x}{x}dx$ .

Let $u = \log_e x$ , $\frac{du}{dx} = \frac{1}{x}$ .

When $x = 1$ , $u = \log_e 1 = 0$ , when $x = e$ , $u = \log_e e = 1$ .

$$\therefore \int_1^e \frac{\log_e x}{x}dx = \int_1^e u\frac{du}{dx}dx = \int_0^1 udu = \left[\frac{1}{2}u^2\right]_0^1 = \frac{1}{2}$$

Example 2 Evaluate $\int_0^{\frac{\pi}{4}} \sec^2 x \tan x dx$ .

Let $u = \tan x$ , $\frac{du}{dx} = \sec^2 x$ .

When $x = 0$ , $u = \tan 0 = 0$ , when $x = \frac{\pi}{4}$ , $u = \tan\frac{\pi}{4} = 1$ .

$$\therefore \int_0^{\frac{\pi}{4}} \sec^2 x \tan x dx = \int_0^{\frac{\pi}{4}} u\frac{du}{dx}dx = \int_0^1 udu = \left[\frac{1}{2}u^2\right]_0^1 = \frac{1}{2}.$$

Example 3 Evaluate $\int_0^{\frac{\pi}{12}} \cos^5(2x)dx$

$\cos^5(2x) = (\cos^2(2x))^2\cos(2x) = (1-\sin^2(2x))^2\cos(2x)$

$= (1-2\sin^2(2x)+\sin^4(2x))\cos(2x)$ .

Let $u = \sin(2x)$ , $\frac{du}{dx} = 2\cos(2x)$ , $\therefore \cos(2x) = \frac{1}{2}\frac{du}{dx}$ .

When $x = 0$ , $u = \sin 0 = 0$ ,

when $x = \frac{\pi}{12}$ , $u = \sin 2\left(\frac{\pi}{12}\right) = \frac{1}{2}$ .

$\therefore \int_0^{\frac{\pi}{12}} \cos^5(2x)dx = \int_0^{\frac{\pi}{12}} (1-2\sin^2(2x)+\sin^4(2x))\cos(2x)dx$

$= \int_0^{\frac{\pi}{12}} (1-2u^2+u^4)\frac{1}{2}\frac{du}{dx}dx$

$= \int_0^{\frac{1}{2}} \frac{1}{2}(1-2u^2+u^4)du$

$= \left[\frac{1}{2}\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right)\right]_0^{\frac{1}{2}}$

$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5\right) - 0$

$= \frac{203}{960}$ .