NSW BOS Mathematics Extension 2 Solutions 2007
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Q1a $\int \frac{1}{\sqrt{9-4x^2}}dx = \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}}dx = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$.

Q1b Let $u = \tan x$, $du = \sec^2 x dx$.
$\int \tan^2 x \sec^2 x dx = \int u^2 du = \frac{1}{3}u^3 + c = \frac{1}{3}\tan^3 x + c$.

Q1c $\frac{d}{dx}(x \sin x) = x\cos x + \sin x$

$\int_0^\pi \frac{d}{dx}(x\sin x)dx = \int_0^\pi (x\cos x + \sin x)dx$

$[x\sin x]_0^\pi = \int_0^\pi x\cos x dx - [\cos x]_0^\pi$

$\int_0^\pi x\cos x dx = \cos\pi - \cos 0 = -2$.

Q1d Let $u = 1 - x$, $x = 1 - u$, $du = -dx$.
When $x = 0$, $u = 1$. When $x = \frac{3}{4}$, $u = \frac{1}{4}$.

$\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}}dx = \int_1^{\frac{1}{4}} -\left(\frac{1-u}{\sqrt{u}}\right)du = \int_{\frac{1}{4}}^1 \frac{1-u}{\sqrt{u}}du = \int_{\frac{1}{4}}^1 \left(u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right)du$

$= \left[2u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_{\frac{1}{4}}^1 = \frac{5}{12}$.

Q1e $\int_{\frac{1}{2}}^2 \frac{2}{x^3+x^2+x+1}dx = \int_{\frac{1}{2}}^2 \left(\frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\right)dx$

$= \left[\log_e(x+1) - \frac{1}{2}\log_e(x^2+1) + \tan^{-1} x\right]_{\frac{1}{2}}^2$

$= \left(\log_e 3 - \frac{1}{2}\log_e 5 + \tan^{-1} 2\right) - \left(\log_e \frac{3}{2} - \frac{1}{2}\log_e \frac{5}{4} + \tan^{-1}\frac{1}{2}\right)$

$= \tan^{-1} 2 - \tan^{-1}\frac{1}{2} = \tan^{-1}\frac{3}{4}$.

Note 1 $\tan\left(\tan^{-1} 2 - \tan^{-1}\frac{1}{2}\right) = \frac{2 - \frac{1}{2}}{1 + 2\times\frac{1}{2}} = \frac{3}{4}$.

Note 2 Let $u = x^2 + 1$, $\frac{1}{2}du = xdx$.

$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log_e u + c = \frac{1}{2}\log_e(x^2+1) + c$.

Q2ai $z = 4 + i$, $w = \bar{z} = 4 - i$.

Q2aii $w - z = (4 - i) - (4 + i) = -2i$.

Q2aiii $\frac{z}{w} = \frac{z^2}{\bar{z}z} = \frac{15+8i}{17} = \frac{15}{17} + \frac{8}{17}i$.

Q2bi $1 + i = 2^{\frac{1}{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

Q2bii $(1+i)^{17} = \left(2^{\frac{1}{2}}\right)^{17}\left(\cos\frac{17\pi}{4} + i\sin\frac{17\pi}{4}\right)$

$= 256\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 256\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 256 + 256i$.

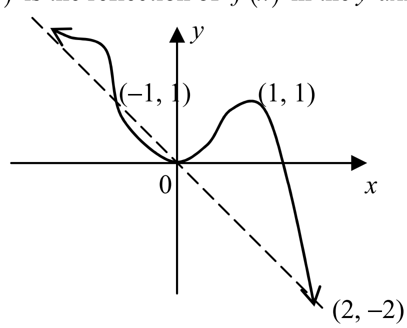
Q2c Let $z = x + yi \neq 0$, $\therefore x \neq 0$ and $y \neq 0$.
$\frac{1}{z} + \frac{1}{\bar{z}} = 1$, $\frac{\bar{z} + z}{z\bar{z}} = 1$, $\frac{2x}{x^2+y^2} = 1$, $2x = x^2 + y^2$,
$\therefore (x-1)^2 + y^2 = 1$, by completing the square.
The locus of P is a circle of radius 1 unit and centred at $(1,0)$, without the point $(0,0)$.

Q2di Let $a = \cos\theta + i\sin\theta$
$z_2 = \cos\left(\theta + \frac{\pi}{3}\right) + i\sin\left(\theta + \frac{\pi}{3}\right)$
$= \left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right][\cos\theta + i\sin\theta] = \omega a$.

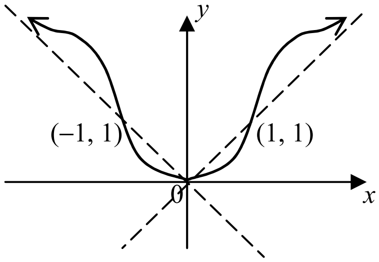
Q2dii $z_1 = \cos\left(\theta - \frac{\pi}{3}\right) + i\sin\left(\theta - \frac{\pi}{3}\right)$
$= \frac{\cos\theta + i\sin\theta}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}} = \frac{a}{\omega}$.
$\therefore z_1 z_2 = \left(\frac{a}{\omega}\right)a\omega = a^2$.

Q2diii $z_1 + z_2 = \frac{a}{\omega} + a\omega = a\left(\frac{1}{\omega} + \omega\right) = a\left(2\cos\frac{\pi}{3}\right) = a$ and
$z_1 z_2 = a^2$, $\therefore z_1$ and z_2 are the roots of $z^2 - az + a^2 = 0$.

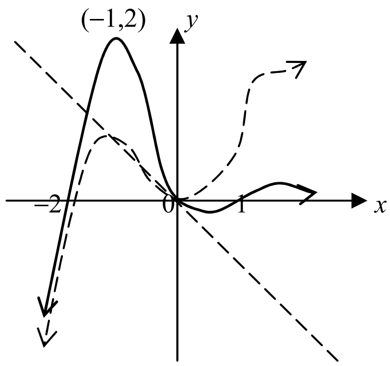
Q3ai $f(-x)$ is the reflection of $f(x)$ in the y-axis.



Q3aii $f(|x|)=\begin{cases} f(x), & x\ge 0 \\ f(-x), & x<0 \end{cases}$



Q3aiii $f(x)-x=f(x)+{}^{-}x$



Q3b The zeros of x^3-5x+3 are α, β and γ.
$\therefore \alpha\beta\gamma=-3$, $\alpha\beta+\beta\gamma+\gamma\alpha=-5$ and $\alpha+\beta+\lambda=0$.
Let x^3+bx^2+cx+d be the cubic with zeros 2α, 2β and 2γ.
$b=-(2\alpha+2\beta+2\gamma)=-2(\alpha+\beta+\lambda)=0$
$c=(2\alpha)(2\beta)+(2\beta)(2\gamma)+(2\gamma)(2\alpha)=4(\alpha\beta+\beta\gamma+\gamma\alpha)=-20$
$d=-(2\alpha)(2\beta)(2\gamma)=-8\alpha\beta\gamma=24$.
Hence the cubic is $x^3-20x+24$.

Q3c $V=\int_1^e 2\pi x\left(\frac{\log_e x}{x}\right)dx=2\pi\int_1^e \log_e x dx$
$=2\pi[x\log_e x-x]_1^e=2\pi[(e-e)-(0-1)]=2\pi$ cubic units.

Note: $\frac{d}{dx}(x\log_e x)=1+\log_e x$, $\frac{d}{dx}(x\log_e x)-1=\log_e x$,
$\int_1^e\left[\frac{d}{dx}(x\log_e x)-1\right]dx=\int_1^e \log_e x dx$,
$\int_1^e \log_e x dx=[x\log_e x-x]_1^e$.

Q3di Vertically, $F\sin\theta+N\cos\theta-mg=0$ ……(1)
Horizontally, $F\cos\theta-N\sin\theta=mr\omega^2$ …………(2)
$\cos\theta\times(1)-\sin\theta\times(2)$, where $\sin\theta\ne 0$ and $\cos\theta\ne 0$:
$N\cos^2\theta+N\sin^2\theta=mg\cos\theta-mr\omega^2\sin\theta$,
$N(\cos^2\theta+\sin^2\theta)=mg\cos\theta-mr\omega^2\sin\theta$,
$N=mg\cos\theta-mr\omega^2\sin\theta$.

Q3dii $N>0$, $mg\cos\theta-mr\omega^2\sin\theta>0$ where $\sin\theta>0$ and $\cos\theta>0$. $\therefore \omega^2<\frac{g\cos\theta}{r\sin\theta}$. Hence $-\sqrt{\frac{g}{r\tan\theta}}<\omega<\sqrt{\frac{g}{r\tan\theta}}$ and $\omega\ne 0$.

Q4a Consider ΔLAM and ΔPAG.
$\angle ALB=\angle APB$, subtended by the same arc on the circumference.
$\angle AMB=\angle AQB$, subtended by the same arc on the circumference.
$\therefore \angle LAM=\angle PAG$.

Q4bi $\sin 3\theta=\sin(2\theta+\theta)=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta$
$=2\sin\theta\cos^2\theta+(\cos^2\theta-\sin^2\theta)\sin\theta$
$=3\sin\theta\cos^2\theta-\sin^3\theta$.

Q4bii $4\sin\theta\sin\left(\theta+\frac{\pi}{3}\right)\sin\left(\theta+\frac{2\pi}{3}\right)$
$=2\sin\theta\left[2\sin\left(\theta+\frac{2\pi}{3}\right)\sin\left(\theta+\frac{\pi}{3}\right)\right]$
$=2\sin\theta\left[\cos\frac{\pi}{3}-\cos(2\theta+\pi)\right]$
$=2\sin\theta\left(\frac{1}{2}+\cos 2\theta\right)$
$=\sin\theta(1+2\cos^2\theta-2\sin^2\theta)$
$=\sin\theta(3\cos^2\theta-\sin^2\theta)$
$=3\sin\theta\cos^2\theta-\sin^3\theta=\sin 3\theta$.

Q4biii $\sin\theta\sin\left(\theta+\frac{\pi}{3}\right)\sin\left(\theta+\frac{2\pi}{3}\right)=\frac{1}{4}\sin 3\theta$.

Max value of $\sin\theta\sin\left(\theta+\frac{\pi}{3}\right)\sin\left(\theta+\frac{2\pi}{3}\right)$ is $\frac{1}{4}$.

Q4c Side length of square $=e-x$.
$V=\int_0^1(e-x)^2dy=\int_0^1(e-e^y)^2dy=\int_0^1(e^2-2e^{y+1}+e^{2y})dy$
$=\left[e^2y-2e^{y+1}+\frac{1}{2}e^{2y}\right]_0^1$
$=\frac{1}{2}(-e^2+4e-1)$ cubic units.

Q4di $(-\alpha)\alpha\beta=-s$, $\therefore \alpha^2\beta=s$
$(-\alpha)\alpha+\alpha\beta+\beta(-\alpha)=r$, $\therefore \alpha^2=-r$
$(-\alpha)+\alpha+\beta=-q$, $\therefore \beta=-q$
$\therefore qr=(-\beta)(-\alpha^2)=\alpha^2\beta=s$

Q4dii Since $r\in R$ and $\alpha^2=-r=i^2r$, $\therefore \alpha=\pm i\sqrt{r}$.
Hence there are two purely imaginary zeros and they are $(-\alpha)=-i\sqrt{r}$ and $\alpha=i\sqrt{r}$.

Q5ai $\Pr(X=3)=\frac{{}^{12}C_3\,{}^{12}C_3}{{}^{24}C_6}=0.36$.

Q5aii $\Pr(X>3)=\frac{{}^{12}C_4\,{}^{12}C_2+{}^{12}C_5\,{}^{12}C_1+{}^{12}C_6\,{}^{12}C_0}{{}^{24}C_6}=0.32$.

Q5bi $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $\therefore \frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$ by implicit differentiation. Hence $\frac{dy}{dx}=\frac{b^2x}{a^2y}$.

At $P(x_1,y_1)$, $m_T=\frac{b^2x_1}{a^2y_1}$. Equation of the tangent is

$y-y_1=\frac{b^2x_1}{a^2y_1}(x-x_1)$, $a^2y_1(y-y_1)=b^2x_1(x-x_1)$,

$b^2x_1x-a^2y_1y=b^2x_1^2-a^2y_1^2$.

Divide both sides by a^2b^2, $\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$.

Since $P(x_1,y_1)$ is on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $\therefore \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$,

$\therefore \frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1$ is the equation of the tangent at $P(x_1,y_1)$

Q5bii Similarly, $\frac{x_2x}{a^2}-\frac{y_2y}{b^2}=1$ is the equation of the tangent at $Q(x_2,y_2)$.

Since $T(x_0,y_0)$ lies on both tangents, $\therefore \frac{x_1x_0}{a^2}-\frac{y_1y_0}{b^2}=1$ and $\frac{x_2x_0}{a^2}-\frac{y_2y_0}{b^2}=1$. Solve these two equations simultaneously to obtain $\frac{y_2-y_1}{x_2-x_1}=\frac{b^2x_0}{a^2y_0}$, which is the gradient of the chord *PQ*.

Equation of *PQ* is $y-y_1=\frac{b^2x_0}{a^2y_0}(x-x_1)$, which can be rearranged to $\frac{x_0x}{a^2}-\frac{y_0y}{b^2}=\frac{x_1x_0}{a^2}-\frac{y_1y_0}{b^2}$. $\therefore \frac{x_0x}{a^2}-\frac{y_0y}{b^2}=1$.

Q5biii Since $S(ae,0)$ lies on $\frac{x_0x}{a^2}-\frac{y_0y}{b^2}=1$, $\therefore \frac{x_0ae}{a^2}=1$,

i.e. $x_0=\frac{a}{e}$. Hence $T(x_0,y_0)$ lies on the directrix $x=\frac{a}{e}$.

Q5ci $(x-1)(5-x)=-x^2+6x-5=2^2-(x-3)^2$.

Q5cii Let $x-3=2\sin\theta$, $dx=2\cos\theta d\theta$,

$\sqrt{(x-1)(5-x)}=\sqrt{2^2-(x-3)^2}=\sqrt{4(1-\sin^2\theta)}=2\cos\theta$.

When $x=1$, $\theta=-\frac{\pi}{2}$. When $x=5$, $\theta=\frac{\pi}{2}$.

$$\int_1^5\sqrt{(x-1)(5-x)}dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4\cos^2\theta d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2(\cos 2\theta+1)d\theta$$

$=[\sin 2\theta+2\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=2\pi$.

Q5di $AD=AC=2AP=2\times 1\cos\frac{\pi}{5}=2u$.

Sum of angles of pentagon $=(n-2)\pi=(5-2)\pi=3\pi$.

$\therefore \angle BAE=\frac{3\pi}{5}$ and $\angle CAD=\frac{1}{2}\left(\pi-\frac{3\pi}{5}\right)=\frac{\pi}{5}$.

Apply the cosine rule to ΔACD,

$CD^2=AC^2+AD^2-2(AC)(AD)\cos\frac{\pi}{5}$.

$\therefore 1^2=(2u)^2+(2u)^2-2(2u)(2u)u$.

$\therefore 8u^3-8u^2+1=0$.

Q5dii Given $x=\frac{1}{2}$ is a root, $\therefore 2x-1$ is a factor of $8x^3-8x^2+1$.

$\therefore 8x^3-8x^2+1=(2x-1)(4x^2-2x-1)=0$.

$\therefore 4x^2-2x-1=0$.

The other roots are $x=\frac{1\pm\sqrt{5}}{4}$.

Hence $u=\frac{1+\sqrt{5}}{4}$, i.e. $\cos\frac{\pi}{5}=\frac{1+\sqrt{5}}{4}$, since $\cos\frac{\pi}{5}>0$.

Q6ai $(a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\ldots+b^n$.

Let $a=b=1$, then $2^n=1+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n-1}+1$.

$\therefore 2^n>\binom{n}{2}$ for $n\ge 2$.

Q6aii $\therefore 2^n>\frac{n(n-1)}{2}$, $2^{n-1}>\frac{n(n-1)}{4}$, $\frac{2^{n-1}}{n+2}>\frac{n(n-1)}{4(n+2)}$,

$\therefore \frac{n+2}{2^{n-1}}<\frac{4n+8}{n(n-1)}$.

Q6aiii It is true for $n=1$ because LHS = RHS = 1.
Assume it is true for $n=k$,

i.e. $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+\ldots+k\left(\frac{1}{2}\right)^{k-1}=4-\frac{k+2}{2^{k-1}}$, then

$1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+\ldots+k\left(\frac{1}{2}\right)^{k-1}+(k+1)\left(\frac{1}{2}\right)^k=4-\frac{k+2}{2^{k-1}}+(k+1)\left(\frac{1}{2}\right)^k$

$=4-\frac{2k+4}{2^k}+\frac{k+1}{2^k}=4-\frac{k+3}{2^k}=4-\frac{(k+1)+2}{2^{(k+1)-1}}$.

$\therefore$ it is also true for $n=k+1$. Hence it is true for all $n\ge 1$.

Q6aiv As $n\to\infty$, $\frac{4n+8}{n(n-1)}=\frac{4+\frac{8}{n}}{n-1}\to 0$.

Since $\frac{n+2}{2^{n-1}}<\frac{4n+8}{n(n-1)}$, $\therefore \frac{n+2}{2^{n-1}}\to 0$

and the series $=4-\frac{n+2}{2^{n-1}}\to 4$.

Q6bi $x = 5\log_e\left(\frac{e^{1.4t}+e^{-1.4t}}{2}\right)$, $2e^{\frac{x}{5}} = e^{1.4t}+e^{-1.4t}$.

Implicit differentiation with respect to *t*:

$\frac{2}{5}e^{\frac{x}{5}}\frac{dx}{dt} = 1.4\left(e^{1.4t}-e^{-1.4t}\right)$, $\therefore \frac{dx}{dt} = 7\left(\frac{e^{1.4t}-e^{-1.4t}}{2e^{\frac{x}{5}}}\right)$,

$\therefore v = 7\left(\frac{e^{1.4t}-e^{-1.4t}}{e^{1.4t}+e^{-1.4t}}\right)$.

Q6bii $2e^{\frac{x}{5}} = e^{1.4t}+e^{-1.4t}$, $\left(2e^{\frac{x}{5}}\right)^2 = \left(e^{1.4t}+e^{-1.4t}\right)^2$,

$\therefore 4e^{\frac{2x}{5}} = e^{2.8t}+e^{-2.8t}+2$.

$v^2 = 49\left(\frac{e^{1.4t}-e^{-1.4t}}{e^{1.4t}+e^{-1.4t}}\right)^2$

$= 49\left(\frac{e^{2.4t}+e^{-2.4t}-2}{4e^{\frac{2x}{5}}}\right) = 49\left(\frac{4e^{\frac{2x}{5}}-4}{4e^{\frac{2x}{5}}}\right) = 49\left(1-e^{\frac{-2x}{5}}\right)$.

Q6biii $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{49}{2}\frac{d}{dx}\left(1-e^{\frac{-2x}{5}}\right) = \frac{49}{5}e^{\frac{-2x}{5}}$

$= \frac{49}{5}\left(1-\frac{v^2}{49}\right) = 9.8-0.2v^2$.

Q6biv $-0.2v^2$ represents the air resistance during the fall. The negative sign shows the direction of the air resistance is opposite to that of motion.

Q6bv Assuming the raindrop has reached terminal velocity, $\ddot{x} = 0$, $9.8-0.2v^2 = 0$, $|v| = 7$ ms^{-1}.

Q7ai For $0 < t < \frac{\pi}{2}$, $\cos t < 1$.

$\therefore \int_0^x \cos t\,dt < \int_0^x 1dt$ where $0 < x < \frac{\pi}{2}$.

$\therefore \sin x < x$ for $0 < x < \frac{\pi}{2}$.

Since $\sin x \le 1$ for all *x*, $\therefore \sin x < x$ for $x > 0$.

Q7aii $f(x) = \sin x - x + \frac{x^3}{6}$, $f'(x) = \cos x - 1 + \frac{x^2}{2}$, $f''(x) = -\sin x + x$. For $x > 0$, $f''(x) = -\sin x + x > 0$, since $\sin x < x$. $\therefore y = f(x)$ is concave up for $x > 0$.

Q7aiii $f''(x) > 0$ for $x > 0$. $\therefore f'(x)$ is an increasing function for $x > 0$, and since $f'(0) = 0$, $f'(x) > 0$ for $x > 0$.

$\therefore f(x)$ is an increasing function, and since $f(0) = 0$, $f(x) > 0$ for $x > 0$. Hence $\sin x > x - \frac{x^3}{6}$ for $x > 0$.

Q7bi ΔPUR and ΔQVR are similar, $\therefore \frac{PU}{QV} = \frac{PR}{QR}$.

Q7bii $QS = eQV$, $PS = ePU$, $\therefore \frac{PU}{QV} = \frac{PS}{QS}$.

Q7biii In ΔPRS, $\frac{PS}{\sin\alpha} = \frac{PR}{\sin(\theta+\phi)}$, $\therefore \frac{PS}{PR} = \frac{\sin\alpha}{\sin(\theta+\phi)}$.

In ΔQRS, $\frac{QS}{\sin\alpha} = \frac{QR}{\sin\theta}$, $\therefore \frac{QR}{QS} = \frac{\sin\theta}{\sin\alpha}$.

$\therefore \frac{QR}{QS}\times\frac{PS}{PR} = \frac{\sin\theta}{\sin\alpha}\times\frac{\sin\alpha}{\sin(\theta+\phi)}$, $\therefore \frac{QR}{PR}\times\frac{PS}{QS} = \frac{\sin\theta}{\sin(\theta+\phi)}$,

$\therefore \frac{QV}{PU}\times\frac{PU}{QV} = \frac{\sin\theta}{\sin(\theta+\phi)}$, $\therefore \frac{\sin\theta}{\sin(\theta+\phi)} = 1$,

i.e. $\sin(\theta+\phi) = \sin\theta$ where $\phi \ne 0$.

Hence $\theta+\phi = \pi-\theta$, i.e. $\phi = \pi - 2\theta$.

Q7biv As $Q \to P$, $\phi \to 0$, $\theta \to \frac{\pi}{2}$.

Q7ci $\frac{PN}{PR} = \cos\beta$, $\frac{ePN}{PR} = e\cos\beta$, $\therefore \frac{PS}{PR} = e\cos\beta$.

Q7cii Similarly, $\frac{PS'}{PW} = e\cos\beta$.

Since $\cos\angle RPS = \frac{PS}{PR}$ and $\cos\angle WPS' = \frac{PS'}{PW}$, $\cos\angle RPS = \cos\angle WPS'$. Hence $\angle RPS = \angle WPS'$.

Q8ai Let $u = a - x$, $du = -dx$. When $x = 0$, $u = a$. When $x = a$, $u = 0$.

$\int_0^a f(a-x)dx = -\int_a^0 f(u)du = \int_0^a f(u)du = \int_0^a f(x)dx$.

Q8aii $f(x) + f(a-x) = f(a)$,

$\int_0^a [f(x)+f(a-x)]dx = \int_0^a f(a)dx$,

$\int_0^a f(x)dx + \int_0^a f(a-x)dx = [xf(a)]_0^a$,

$2\int_0^a f(x)dx = af(a)$, $\therefore \int_0^a f(x)dx = \frac{a}{2}f(a)$.

Q8bi The series is geometric, $a = 1$, $r = z^2$.

$S_n = \frac{(z^2)^n - 1}{z^2-1} = \frac{z^{2n}-1}{z^2-1} = \frac{z^{2n-1}-z^{-1}}{z-z^{-1}} = \left(\frac{z^n - z^{-n}}{z-z^{-1}}\right)z^{n-1}$.

Q8bii Let $z = \cos\theta + i\sin\theta$,

$z^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$,

$z^2 = \cos 2\theta + i\sin 2\theta$, …,

$z^{2n-2} = \cos(2n-2)\theta + i\sin(2n-2)\theta$,

$z^n = \cos n\theta + i\sin n\theta$, $z^{n-1} = \cos(n-1)\theta + i\sin(n-1)\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$.

$\text{LHS} = 1 + \cos 2\theta + ... + \cos(2n-2)\theta + i[\sin 2\theta + ... + \sin(2n-2)\theta]$

$$\text{RHS} = \left(\frac{(\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)}{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}\right) \times [\cos(n-1)\theta + i\sin(n-1)\theta]$$

$$= \frac{\sin n\theta}{\sin\theta}[\cos(n-1)\theta + i\sin(n-1)\theta].$$

Q8biii Equate the imaginary parts on both sides and let $\theta = \frac{\pi}{2n}$.

$$\text{LHS} = \sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + ... + \sin\frac{(n-1)\pi}{n}.$$

$$\text{RHS} = \frac{\sin\frac{\pi}{2}}{\sin\frac{\pi}{2n}}\sin\frac{(n-1)\pi}{2n} = \frac{1}{\sin\frac{\pi}{2n}}\sin\left(\frac{\pi}{2} - \frac{\pi}{2n}\right) = \frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}} = \cot\frac{\pi}{2n}.$$

Q8ci
$$d_1 + ... + d_{n-1} = \frac{\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}} + \frac{\sin\frac{2\pi}{n}}{\sin\frac{\pi}{n}} + ... + \frac{\sin\frac{(n-1)\pi}{n}}{\sin\frac{\pi}{n}}$$

$$= \frac{\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + ... + \sin\frac{(n-1)\pi}{n}}{\sin\frac{\pi}{n}} = \frac{\cot\frac{\pi}{2n}}{\sin\frac{\pi}{n}} = \frac{\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}}{2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}$$

$$= \frac{1}{2\sin^2\frac{\pi}{2n}}.$$

Q8cii $p = n$, $q = \frac{1}{n}(d_1 + ... + d_{n-1})$,

$$\therefore \frac{p}{q} = \frac{n^2}{d_1 + ... + d_{n-1}} = n^2\left(2\sin^2\frac{\pi}{2n}\right) = 2\left(n\sin\frac{\pi}{2n}\right)^2.$$

Q8ciii
$$\frac{p}{q} = \frac{n^2}{d_1 + ... + d_{n-1}} = 2\left(n\sin\frac{\pi}{2n}\right)^2 = \frac{\pi^2}{2}\left(\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\right)^2.$$

As $n \to \infty$, $\frac{\pi}{2n} \to 0$, $\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}} \to 1$, $\therefore \frac{p}{q} \to \frac{\pi^2}{2}$.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.