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VCAA Mathematical Methods 34
Sample exam 2 solutions 2006

SECTION 1: Multiple-choice questions

1	2	3	4	5	6	7	8	9	10	11
C	B	A	D	A	A	B	D	D	A	A

12	13	14	15	16	17	18	19	20	21	22
A	C	A	C	D	A	B	B	A	C	E

Q1 $2\sin(3x)-1=0,\ \sin(3x)=\frac{1}{2},\ 3x=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6}$

$x=\frac{\pi}{18},\frac{5\pi}{18},\frac{13\pi}{18},\frac{17\pi}{18}$. $\frac{\pi}{18}+\frac{5\pi}{18}+\frac{13\pi}{18}+\frac{17\pi}{18}=2\pi.$

Q2 $e^{2x}-3e^{x}+2=0,\ (e^{x}-1)(e^{x}-2)=0,\ \therefore e^{x}=1$ or $e^{x}=2.$
Hence $x=0$ or $x=\log_e 2.$

Q3 $|p+3|>3,$ either $p+3>3$ or $-(p+3)>3.$
$\therefore p>0$ or $p+3<-3,\ \therefore p>0$ or $p<-6.$

Q4 Area $=\int_a^b f(x)dx-\int_b^c f(x)dx=\int_a^b f(x)dx+\int_c^b f(x)dx$.

Q5 Use graphics calculator, $h=0.5\left(1-e^{-0.05t}\cos\left(\frac{3\pi t}{2}\right)\right)$,

evaluate $\frac{dh}{dt}$ at $t=2.5$. $\frac{dh}{dt}=0.03$.

By calculus, $\frac{dh}{dt}=0.5\left(\frac{3\pi}{2}e^{-0.05t}\sin\left(\frac{3\pi t}{2}\right)+0.05e^{-0.05t}\cos\left(\frac{3\pi t}{2}\right)\right)$.

At $t=2.5$, $\frac{dh}{dt}=0.03$.

Q6 $N(t)=1000e^{0.1t}$, $N(0)=1000,\ N(10)=1000e.$
Average rate $=\frac{1000e-1000}{10}=172$.

Q7 $y=a(x+3)(x-1)(x-4),$ when $x=0,\ y=24,$
$\therefore 24=a(3)(-1)(-4),\ \therefore a=2.$

Q8 $\log_5(6)=\frac{\log_{10}(6)}{\log_{10}(5)}=1.113$

Q9 The graph is the inverse of $y=x^3$, $\therefore y=\sqrt[3]{x}=x^{\frac{1}{3}}$ is the rule of the graph.

Q10 $f(x)=\log_e(x^2)+1$ is defined for $x\neq 0$. D is $R\setminus\{0\}$.

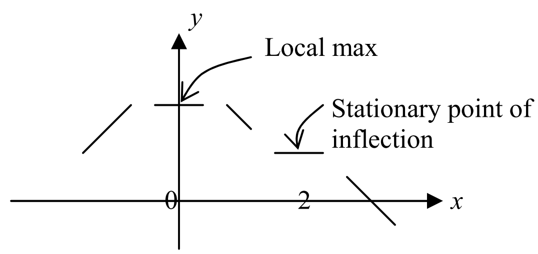
Q11 $f(x)=2x^3-3x^2+6,$ to find the stationary points let
$f'(x)=6x^2-6x=0$, $\therefore 6x(x-1)=0,\ \therefore x=0$ or $x=1$.
For *f* to have an inverse function, *f* must be a one-to-one function, $\therefore a\geq 1$.

Q12 $\int_a^b(4-2f(x))dx=\int_a^b 4dx-2\int_a^b f(x)dx=[4x]_a^b-2(3)$
$=4b-4a-6=4(b-a)-6$.

Q13 Amplitude is doubled (dilation from the x-axis by a scale factor of 2) and the graph is inverted (reflection in the x-axis).

Q14 For $\pi<x<2\pi,\ \sin(x)$ has a negative value,
$\therefore y=|\sin(x)|=-\sin(x)$ and $\frac{dy}{dx}=-\cos(x)$.
Hence at $x=k,\ \frac{dy}{dx}=-\cos(k)$.

Q15



Q16 For $t\in[a,0),\ \int_t^0 f(x)dx>0$ because $f(x)>0.$
$\therefore F(t)=\int_0^t f(x)dx=-\int_t^0 f(x)dx<0$.
For $t=0,\ F(t)=\int_0^t f(x)dx=\int_0^0 f(x)dx=0.$
For $t\in(0,b],\ F(t)=\int_0^t f(x)dx>0$ because $f(x)>0.$

Q17 At $x=4,\ y=2x^{\frac{3}{2}}=2\left(4^{\frac{3}{2}}\right)=16,$

gradient of tangent $=\frac{dy}{dx}=3x^{\frac{1}{2}}=6,$ gradient of normal $=-\frac{1}{6}.$

Equation of normal: $y-16=-\frac{1}{6}(x-4),$

$\therefore y=-\frac{1}{6}x+\frac{50}{3}.$

Q18 $f(x)$ is an increasing function, $\therefore f'(x)>0$ for all *x*. Also as x increases, $f'(x)$ increases.

Q19 $\Pr(X>15)=\Pr\left(Z>\frac{15-12.2}{1.4}\right)=\Pr(Z.>2)$.

Q20 Given $p = 0.15$, and $\Pr(X \geq 1) > 0.95$.
$\therefore 1 - \Pr(X = 0) > 0.95$, $\therefore \Pr(X = 0) < 0.05$,
$\therefore (1 - 0.15)^n < 0.05$, $0.85^n < 0.05$,
$n \log_{10} 0.85 < \log_{10} 0.05$, $-0.0706n < -1.3010$,
$\therefore n > 18.4$.

Q21 $\Pr(00 \cup 11 \cup 22 \cup 33) = \Pr(00) + \Pr(11) + \Pr(22) + \Pr(33)$
$= 0.4^2 + 0.3^2 + 0.2^2 + 0.1^2 = 0.30$

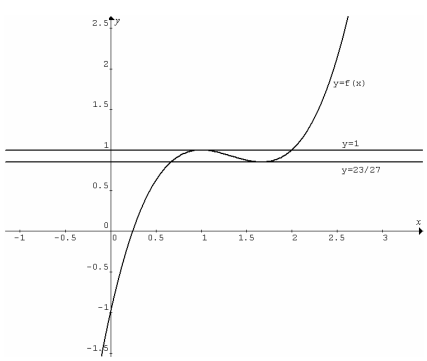
Q22
$\Pr(X > a) = 0.25$, $\int_a^{\pi} \left(\frac{1}{2}\sin(x)\right) dx = 0.25$, $\left[-\frac{1}{2}\cos(x)\right]_a^{\pi} = 0.25$,
$-\frac{1}{2}\cos(\pi) + \frac{1}{2}\cos(a) = 0.25$, $1 + \cos(a) = 0.5$, $\cos(a) = -0.5$,
$\therefore a \approx 2.09$.

SECTION 2

Q1a $f(x) = (x-1)^2(x-2) + 1$,
$f'(x) = (x-1)^2(1) + 2(x-1)(x-2) = (x-1)((x-1) + 2(x-2))$
$= (x-1)(3x-5)$. $\therefore u = 3$ and $v = -5$.

Q1b At the turning points, $f'(x) = 0$, $(x-1)(3x-5) = 0$,
$x = 1$ and $y = 1$ or $x = \frac{5}{3}$ and $y = \frac{23}{27}$.
$\therefore a = 1$ and $b = \frac{5}{3}$.

Q1c



For $(x-1)^2(x-2) + 1 = p$ to have exactly one solution, $p < \frac{23}{27}$ or $p > 1$.

Q1di Dilation from the y-axis by a scale factor of 2, then downward translation by 1 unit.

Q1dii $y = f\left(\frac{x}{2}\right) - 1 = \left(\frac{x}{2} - 1\right)^2\left(\frac{x}{2} - 2\right) + 1 - 1 = \left(\frac{x}{2} - 1\right)^2\left(\frac{x}{2} - 2\right)$,
x-intercepts: $\left(\frac{x}{2} - 1\right)^2\left(\frac{x}{2} - 2\right) = 0$, $\therefore \frac{x}{2} - 1 = 0$ or $\frac{x}{2} - 2 = 0$.
Hence $x = 2$ or 4.

Q1diii Use graphics calculator to sketch $y = f\left(\frac{x}{2}\right) - 1$ and evaluate definite integral (lower limit 2, upper limit 4) to obtain 0.17.

Q1e $f(x+h) = 1$, $(x+h-1)^2(x+h-2) + 1 = 1$,
$\therefore (x+h-1)^2(x+h-2) = 0$.
To have exactly one positive value solution, $h - 1 \geq 0$ and $h - 2 \leq 0$. $\therefore h \geq 1$ and $h \leq 2$, i.e. $1 \leq h \leq 2$.
Note: zero is neither positive nor negative.

Q2a $y = (2x^2 - 3x)e^{ax}$ passes through $(2,3)$.
$3 = (2(2^2) - 3(2))e^{2a}$, $3 = 2e^{2a}$, $\therefore 2a = \log_e 1.5$, $a = 0.203$.

Q2b $a = 1$, $\therefore y = (2x^2 - 3x)e^x = x(2x-3)e^x$.
x-intercepts: $x = 0$ or $\frac{3}{2}$. The x-coordinate of A is $\frac{3}{2}$.

Q2ci $y = (2x^2 - 3x)e^x$,
$\frac{dy}{dx} = (4x-3)e^x + (2x^2 - 3x)e^x = e^x(2x^2 + x - 3)$
$\therefore p = 2$, $q = 1$, $r = -3$.

Q2cii At turning points, $\frac{dy}{dx} = 0$,
$\therefore e^x(2x^2 + x - 3) = e^x(2x+3)(x-1) = 0$.
Since $e^x \neq 0$, $\therefore x = -\frac{3}{2}$ or 1. Hence x-coordinate of B is 1 and y-coordinate is $-e$. $\therefore$B is $(1.000, -2.718)$.

Q2d $\frac{d}{dx}\{(2x^2 + mx + n)e^x\} = (2x^2 - 3x)e^x$,
$\therefore (4x+m)e^x + (2x^2 + mx + n)e^x = (2x^2 - 3x)e^x$,
$\therefore 2x^2 + (m+4)x + (m+n) = 2x^2 - 3x$,
$\therefore m + 4 = -3$ and $m + n = 0$.
Hence $m = -7$ and $n = 7$.
$\text{Area} = -\int_0^{1.5}(2x^2 - 3x)e^x dx = -\left[(2x^2 + mx + n)e^x\right]_0^{1.5}$
$= -\left[(2x^2 - 7x + 7)e^x\right]_0^{1.5}$
$= -(2(1.5)^2 - 7(1.5) + 7)e^{1.5} + (2(0)^2 - 7(0) + 7)e^0$
$= 7 - e^{1.5}$.

Q3ai Given Monday night on north side,
$\Pr(NNN) = \left(\frac{2}{5}\right)^3 = \frac{8}{125}$.

Q3aii Given Monday night on north side,
$\Pr(NNS) + \Pr(NSN) + \Pr(SNN)$
$= \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} + \frac{2}{5} \times \frac{3}{5} \times \frac{4}{5} + \frac{3}{5} \times \frac{4}{5} \times \frac{2}{5} = \frac{12}{25}$.

Q3aiii Given Monday night on north side,
$\Pr(NNS) + \Pr(NNNS)$
$= \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} + \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} = \frac{84}{625}$.

Q3b $\Pr(t>3)=\int_3^4\left(\frac{3}{32}t(4-t)\right)dt=\left[\frac{3}{32}\left(2t^2-\frac{t^3}{3}\right)\right]_3^4$

$=\frac{3}{32}\left(2(4^2)-\frac{4^3}{3}\right)-\frac{3}{32}\left(2(3^2)-\frac{3^3}{3}\right)=\frac{5}{32}.$

Q3c Binomial, $p=\frac{5}{32},\ n=3.$

$\Pr(X\geq 2)=1-\Pr(X\leq 1)=1-0.934=0.066$.

Q3d $\Pr\left(t<\frac{n}{60}\right)=\int_0^{\frac{n}{60}}\left(\frac{3}{32}t(4-t)\right)dt=\left[\frac{3}{32}\left(2t^2-\frac{t^3}{3}\right)\right]_0^{\frac{n}{60}}$

$=\frac{3}{32}\left(2\left(\frac{n}{60}\right)^2-\frac{1}{3}\left(\frac{n}{60}\right)^3\right)=\frac{1}{32}\left(\frac{n^2}{600}-\frac{n^3}{216000}\right).$

$\therefore \frac{1}{32}\left(\frac{n^2}{600}-\frac{n^3}{216000}\right)=0.104$. Use graphics calculator to solve this equation. $n=48$.

Q4a Since $-1\leq\sin\left(\frac{(5t-1)\pi}{2}\right)\leq 1$

$\therefore$ max height = 62 + 60 = 122

Q4b min height = 62 – 60 = 2

Q4c Period $T=\frac{2\pi}{n}=\frac{2\pi}{\frac{5\pi}{2}}=0.8$ hour i.e. 48 minutes.

$\therefore$ At 1.48 pm.

Q4di $92=62+60\sin\left(\frac{(5t-1)\pi}{2}\right)$, $\therefore \sin\left(\frac{(5t-1)\pi}{2}\right)=\frac{1}{2}$,

$\frac{(5t-1)\pi}{2}=\frac{\pi}{6},\frac{5\pi}{6}$.

Hence $t=\frac{4}{15}$ (first time), $\frac{8}{15}$ (second time).

$t=\frac{4}{15}$ hour $=16$ minutes, $\therefore$ at 1.16 pm.

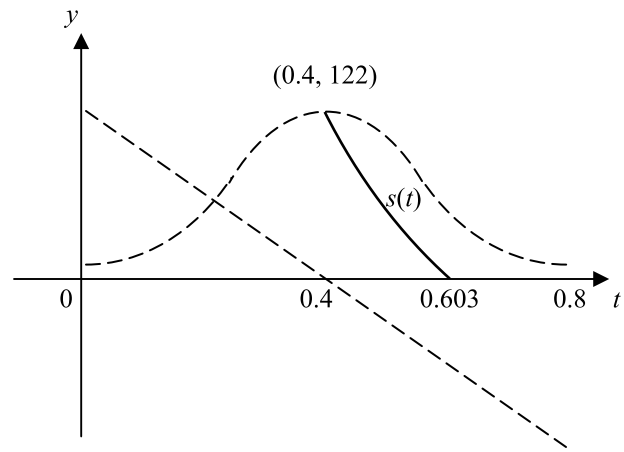
Q4dii At least 92 metres above ground level when $\frac{4}{15}\leq t\leq\frac{8}{15}$,

$\therefore \Delta t=\frac{8}{15}-\frac{4}{15}=\frac{4}{15}$ hour, i.e. 16 minutes.

Q4ei $\frac{dh}{dt}=60\times\frac{5\pi}{2}\cos\left(\frac{(5t-1)\pi}{2}\right)=150\pi\cos\left(\frac{(5t-1)\pi}{2}\right)$.

Q4eii When $t=1,\frac{dh}{dt}=150\pi\cos(2\pi)=471.2$ mh^{-1}.

Q4f i and ii



Domain for $s(t)$ is $[0.4, 0.603]$.

↑

Graphics calculator

Q4fiii Spider reaches ground when $s(t)=0$,

i.e. at $t=0.603$ hour.

At the highest point, $t=0.4$ hour.

$\therefore \Delta t=0.603-0.4=0.203$ hour, i.e. 12 minutes.

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