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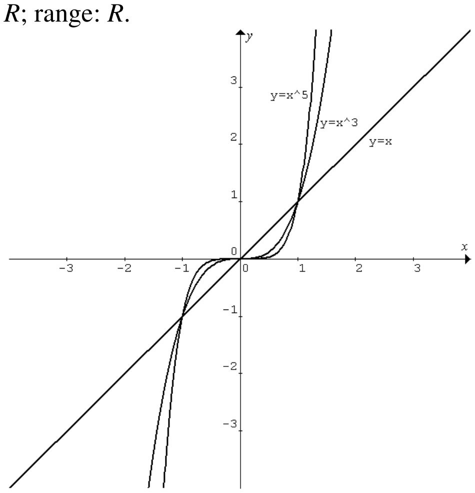
Functions and graphs

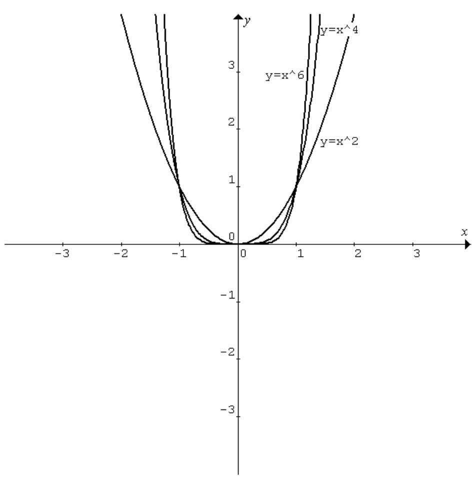
Power functions $y = x^n$, for $n \in Q$ (set of rational numbers)

The graph of $y = x$ is a straight line through the origin $(0,0)$.
Domain: *R*; range: *R*.

The graphs of even-power functions, e.g. $y = x^2$ and $y = x^4$ have a turning-point at $(0,0)$. They show symmetry under a reflection in the *y*-axis. $\therefore$ the *y*-axis, i.e. the line $x = 0$ is called the axis of symmetry.
Domain: *R*; range: $[0,\infty)$.

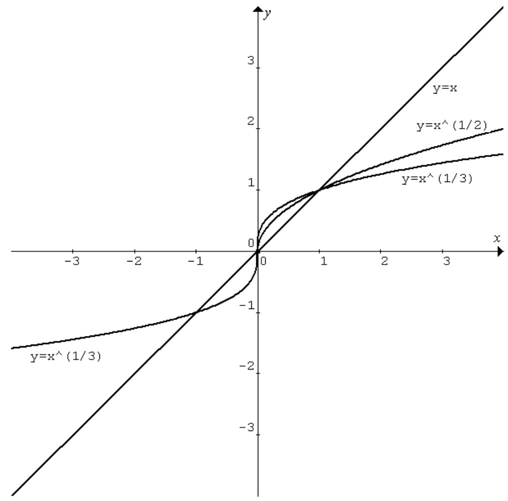
The graphs of odd-power functions, e.g. $y = x^3$ and $y = x^5$ have a stationary point of inflection at $(0,0)$.
Domain: *R*; range: *R*.





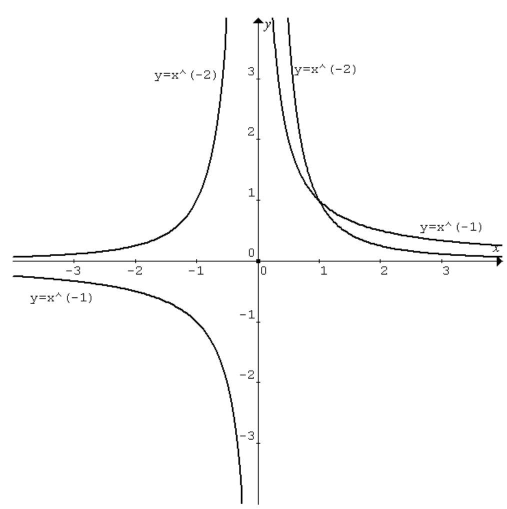
The function $y = x^{\frac{1}{2}}$ can be written as $y = \sqrt{x}$. It is undefined for $x < 0$. It has an end point at $(0,0)$.
Domain: $[0,\infty)$; range: $[0,\infty)$.

The function $y = x^{\frac{1}{3}}$ can be expressed as $y = \sqrt[3]{x}$. It has a vertical tangent at $(0,0)$.
Domain: *R*; range: *R*.



The graph of $y = x^{-1}$ (or $y = \frac{1}{x}$) consists of two branches, one in the first quadrant and the other in the third quadrant. The axes of symmetry are lines $y = \pm x$. The function shows the following asymptotic behaviours:
As $x \to -\infty$, $y \to 0^-$; as $x \to +\infty$, $y \to 0^+$. $\therefore y = 0$ is the horizontal asymptote of the function.
As $x \to 0^-$, $y \to -\infty$; as $x \to 0^+$, $y \to +\infty$. $\therefore x = 0$ is the vertical asymptote.
Domain: $R\backslash\{0\}$; range: $R\backslash\{0\}$.

The graph of $y = x^{-2}$ (or $y = \frac{1}{x^2}$) also has two branches, one in the first quadrant and the other in the second quadrant. The line $x = 0$ is the axis of symmetry. The function shows the following asymptotic behaviours:
As $x \to -\infty$, $y \to 0^+$; as $x \to +\infty$, $y \to 0^+$. $\therefore y = 0$ is the horizontal asymptote of the function.
As $x \to 0^-$, $y \to +\infty$; as $x \to 0^+$, $y \to +\infty$. $\therefore x = 0$ is the vertical asymptote.
Domain: $R\backslash\{0\}$; range: R^+.

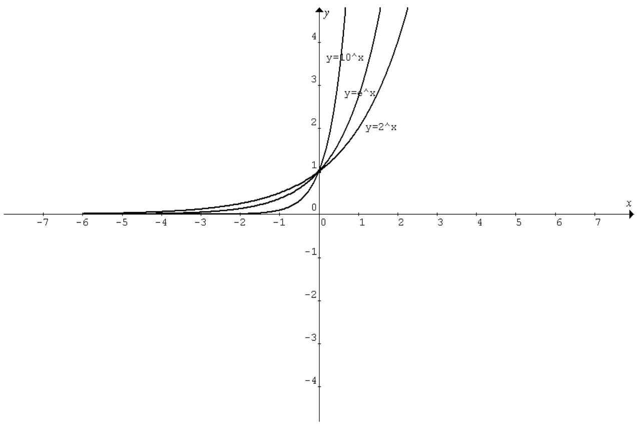


Exponential functions $y = a^x$, where $a \in R^+$

For $a > 1$, the graphs of functions with equation $y = a^x$ have the same shape and the same *y*-intercept $(0,1)$.

Asymptotic behaviour: As $x \to -\infty$, $y \to 0^+$, the same horizontal asymptote $y = 0$ for the functions.

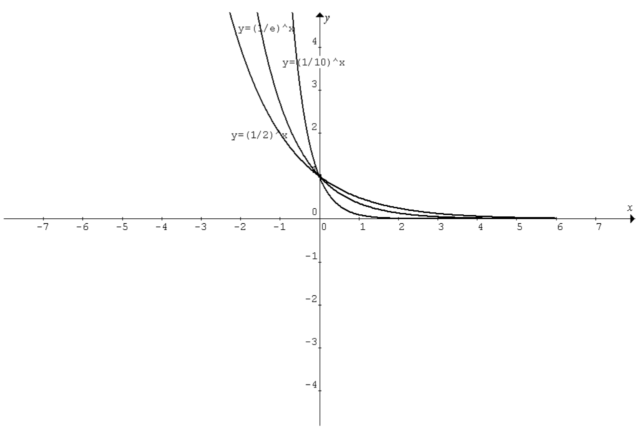
a^x is always > 0.
Domain: *R*; range: R^+.



For $0 < a < 1$, the graphs of $y = a^x$ have the same shape and the same *y*-intercept $(0,1)$.

Asymptotic behaviour: As $x \to +\infty$, $y \to 0^+$, the same horizontal asymptote $y = 0$ for the functions.

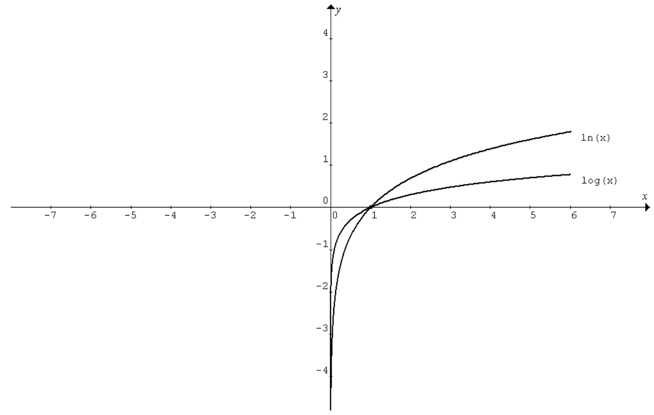
a^x is always > 0.
Domain: *R*; range: R^+.



Logarithmic functions $y = \log_e x$ and $y = \log_{10} x$

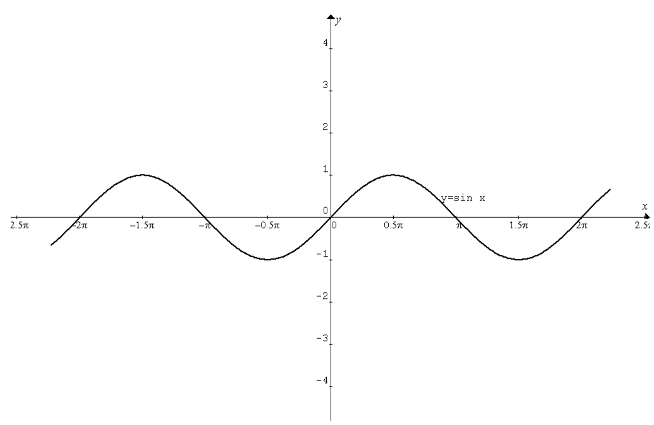
The log functions $y = \log_e x$ ($\ln(x)$ on calculators) and $y = \log_{10} x$ ($\log(x)$ on calculators) have the same shape and a common *x*-intercept $(1,0)$. They are undefined for $x \le 0$. Both functions have a negative value for $0 < x < 1$ and a positive value for $x > 1$.

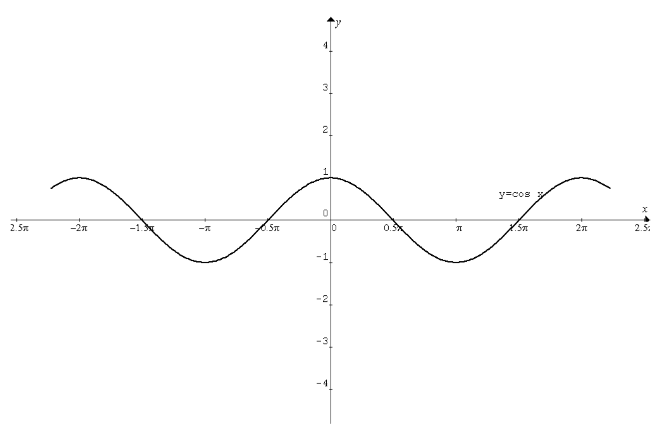
Asymptotic behaviour: As $x \to 0^+$, $y \to -\infty$, the same vertical asymptote $x = 0$.
Domain: R^+; range: *R*.



Circular (trigonometric) functions $y = \sin x$, $y = \cos x$

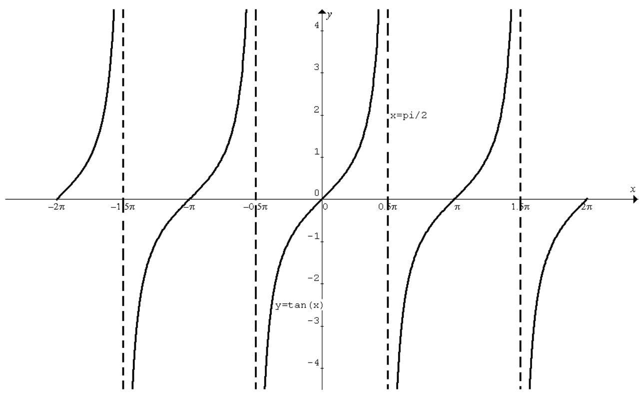
Both functions are periodic functions. Each repeats itself after a period of $T = 2\pi$, i.e. each has symmetry property under a horizontal translation of 2π, e.g. $\sin(a \pm 2\pi) = \sin a$, $\cos(a \pm 2\pi) = \cos a$.
Other symmetry properties:
For $\cos x$, under a reflection in the *y*-axis, $\cos(-a) = \cos a$.
For $\sin x$, under a reflection in the *y*-axis and a horizontal translation of π, $\sin(\pi - a) = \sin a$; under combine reflections in both axes, $-\sin(-a) = \sin a$.
The value of each function fluctuates between −1 and 1 inclusively, the *amplitude* of each is 1.
Domain: *R*; range: $[-1,1]$.





Circular (trigonometric) function $y = \tan x$

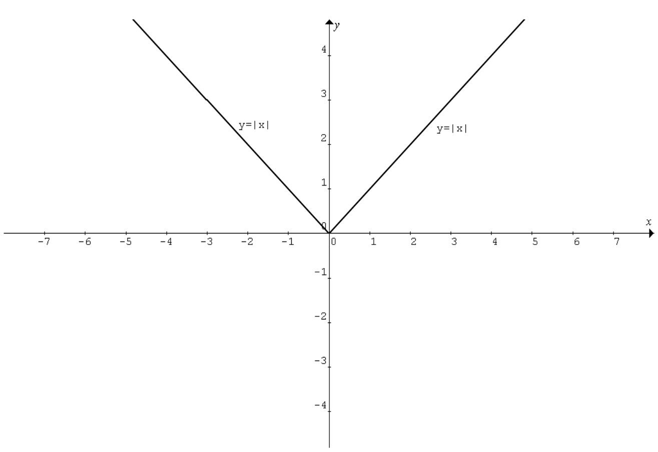
The function $y = \tan x$ is also a periodic function. It repeats itself after a period of $T = \pi$, i.e. it has symmetry property under a horizontal translation of π, $\tan(a \pm \pi) = \tan a$.
It also has symmetry property under combine reflections in both axes, $-\tan(-a) = \tan a$.
The term *amplitude* is not applicable for $y = \tan x$.
The function is undefined at $x = \pm\left(n - \frac{1}{2}\right)\pi$ for $n \in J^+$ (set of positive integers). It shows asymptotic behaviour as $x \to \pm\left(n - \frac{1}{2}\right)\pi$. The vertical asymptotes are $x = \pm\left(n - \frac{1}{2}\right)\pi$.
Domain: $\left\{x : x \neq \pm\left(n - \frac{1}{2}\right)\pi, n \in J^+\right\}$; range: *R*.



Modulus function $y = |x|$

$y = |x|$ can be defined as $y = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$ or $y = \sqrt{x^2}$.

It has symmetry property under reflection in the *y*-axis, i.e. $y = |x|$ and $y = |-x|$ are the same, and the line $x = 0$ is the axis of symmetry. Its vertex is $(0,0)$.
Domain: *R*; range: $[0, \infty)$.



Transformations

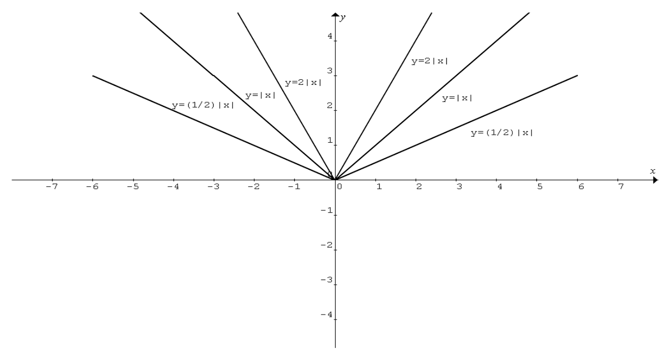
Any of the above functions can be transformed by one or a combination of the following function operations.

Vertical dilation of function $y = f(x)$:

$y = f(x) \to y = Af(x)$, where $A > 0$

For $0 \leq A < 1$, the graph of $y = f(x)$ is compressed toward the *x*-axis to give it a wider appearance; for $A > 1$, it is stretched away from the *x*-axis to give it a narrower appearance. *A* is called the dilation factor.
Other ways to say vertical dilation are:
- dilation parallel to the *y*-axis
- dilation from (or toward) the *x*-axis

Example Compare the graphs of the transformed functions $y = \frac{1}{2}|x|$ and $y = 2|x|$ with the graph of the original function $y = |x|$.

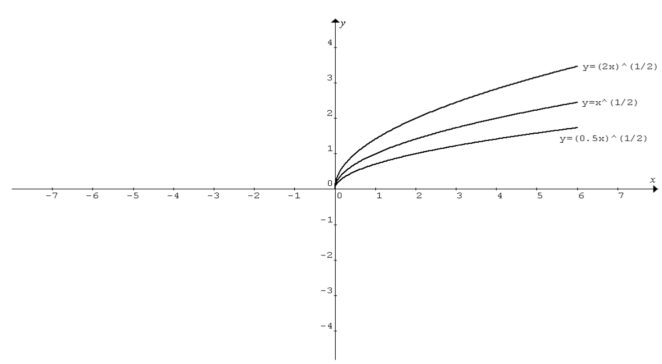


Horizontal dilation of function $y = f(x)$:

$y = f(x) \to y = f(nx)$, where $n > 0$

For $0 < n < 1$, the graph of $y = f(x)$ is stretched away from the *y*-axis to give it a wider appearance; for $n > 1$, it is compressed toward the *y*-axis to give it a narrower appearance. The dilation factor is $\frac{1}{n}$ for this transformation.
Other ways to say horizontal dilation are:
- dilation parallel to the *x*-axis
- dilation from (or toward) the *y*-axis

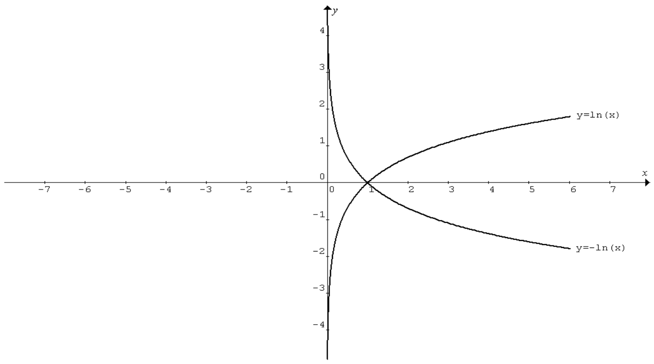
Example Compare $y = \sqrt{0.5x}$ and $y = \sqrt{2x}$ with $y = \sqrt{x}$.



Reflection of function $y = f(x)$ *in the x-axis*:

$y = f(x) \rightarrow y = -f(x)$

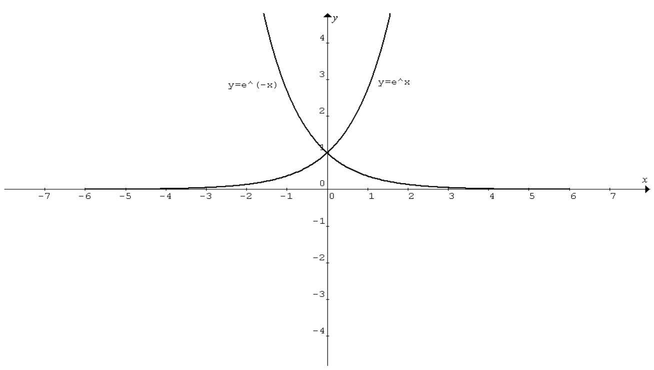
Example Compare $y = -\log_e x$ with $y = \log_e x$.



Reflection of function $y = f(x)$ *in the y-axis*:

$y = f(x) \rightarrow y = f(-x)$

Example Compare $y = e^{-x}$ with $y = e^x$.

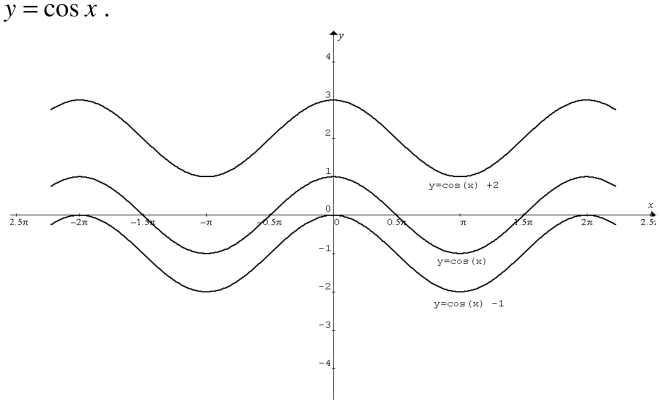


Vertical translation of function $y = f(x)$ *by c units*:

$y = f(x) \rightarrow y = f(x) \pm c$, where $c > 0$

The + and – operations correspond to upward and downward translations respectively.

Example Compare $y = \cos x + 2$ and $y = \cos x - 1$ with $y = \cos x$.

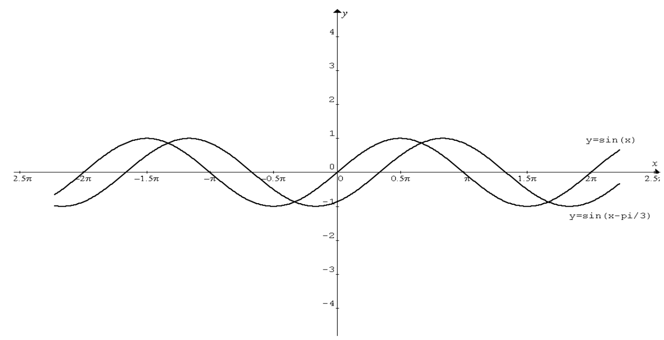


Horizontal translation of function $y = f(x)$ *by b units*:

$y = f(x) \rightarrow y = f(x \pm b)$, where $b > 0$

The + and – operations correspond to left and right translations respectively.

Example Compare $y = \sin\left(x - \frac{\pi}{3}\right)$ with $y = \sin x$.



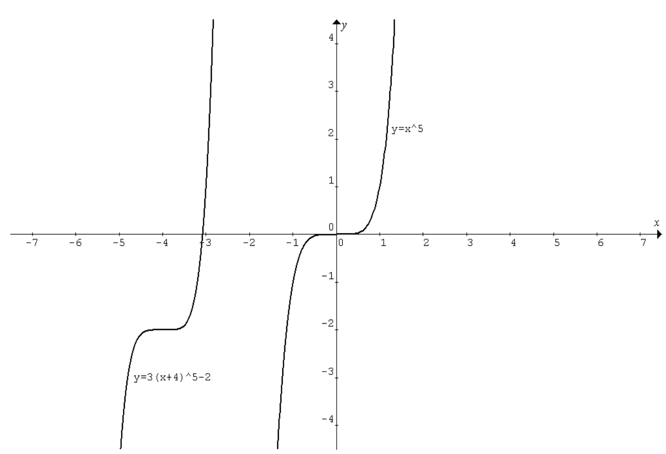
Combination of transformations

If a transformed function is the result of a combination of the above transformations, it would be easier to recognise the transformations involved by expressing the function in the form $y = \pm A\ f(\pm n(x \pm b)) \pm c$.

- vertical translation (+ up)
- horizontal translation (+ left)
- horizontal dilation (factor 1/*n*)
- reflection in the *y*-axis (– sign)
- vertical dilation (factor *A*)
- reflection in the *x*-axis (– sign)

To sketch the transformed function from the original function, always carry out translations last.

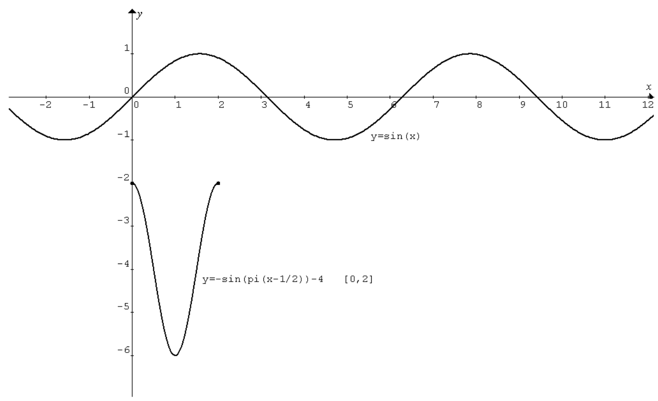
Example 1 Sketch $y = 3(x+4)^5 - 2$.
This function is the result of a combination of transformations of $y = x^5$. It involves a vertical dilation by a factor of 3, and translations of 4 left and 2 down. The stationary point of inflection changes from $(0,0)$ to $(-4,-2)$.



Example 2 Sketch $y=-2\sin\left(\pi x-\frac{\pi}{2}\right)-4$ for $0\le x\le 2$.

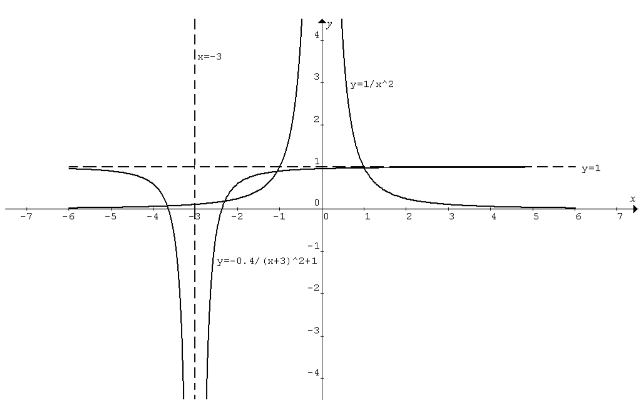
Express the function as $y=-2\sin\pi\left(x-\frac{1}{2}\right)-4$.

This function is the result of a combination of transformations of $y=\sin x$. The amplitude changes from 1 to 2 (note: not -2) and the period changes from 2π to $T=\frac{2\pi}{\pi}=2$. They correspond to a vertical dilation of the function by a factor of 2 and a horizontal dilation by a factor of $\frac{1}{\pi}$ respectively.
There is a reflection in the x-axis followed by translations of $\frac{1}{2}$ right and 4 down.



Example 3 Sketch $y=\frac{-0.4}{(x+3)^2}+1$.

This function is the result of a combination of transformations of $y=\frac{1}{x^2}$. It involves a reflection in the x-axis and a vertical dilation by a factor of 0.4, and then translations 3 left and 1 up. The function has $x=-3$ and $y=1$ as its asymptotes.



Example 4 Sketch $y=-\left|1-\frac{x}{2}\right|+1$.

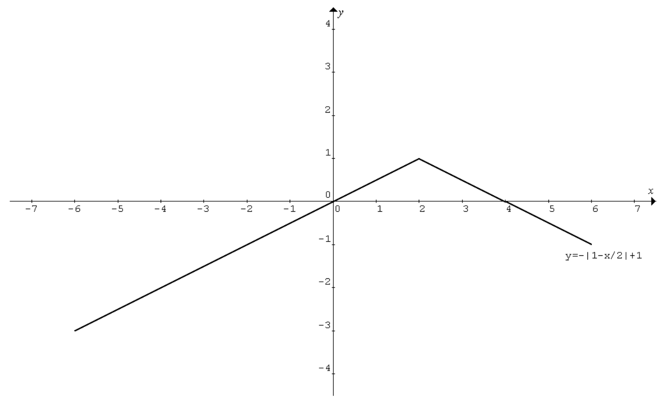
This function is the result of a combination of transformations of $y=|x|$.

Firstly express it in the form $y=\pm Af(\pm n(x\pm b))\pm c$.

$y=-\frac{1}{2}|2-x|+1=-\frac{1}{2}|-(x-2)|+1=-\frac{1}{2}|x-2|+1$.

The transformations are:
- reflection in the x-axis
- vertical dilation by factor $\frac{1}{2}$
- translations 2 right and 1 up.

The vertex is $(2,1)$.



Polynomial functions

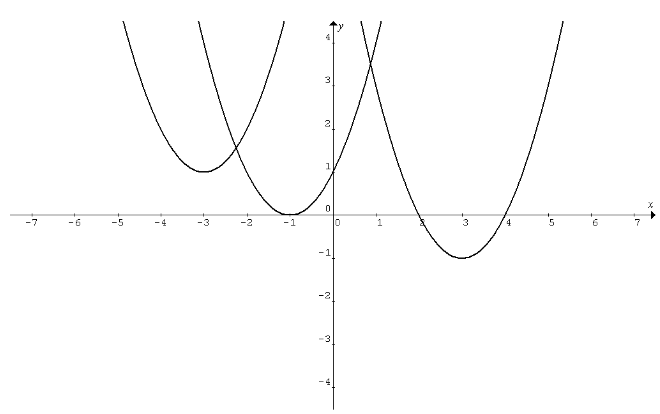
A polynomial function P of x is a linear combination of power functions x^n, where $n\in\{0,1,2,3,......\}$.

Examples are:

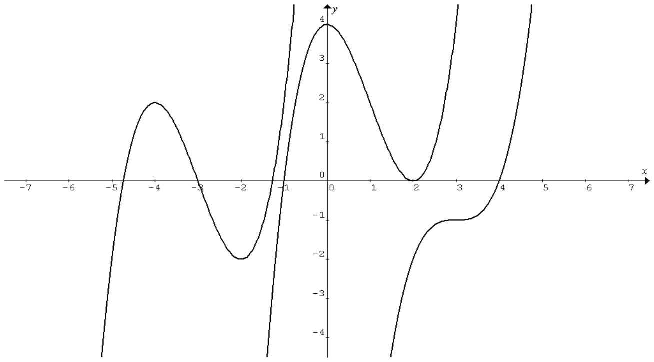
$P(x)=2x-5$	(a linear function)
$P(x)=-3x^2+x+2$	(a quadratic function)
$P(x)=0.2x^3-\frac{x^2}{3}+\left(\sqrt{5}\right)x-\pi$	(a cubic function)
$P(x)=-x^4+\left(\sqrt[3]{4}\right)x-e^2$	(a quartic function)

Some polynomial functions can be changed to factorised form. The *linear* factors give the x-intercepts. Some polynomial functions may not have any linear factors; hence not all polynomial functions have x-intercepts.

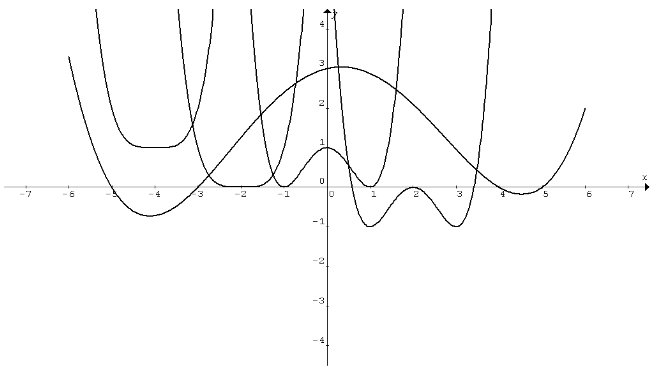
A quadratic function may have 0, 1 or 2 distinct linear factors, hence 0, 1 or 2 x-intercepts.



A cubic function may have 1, 2 or 3 distinct linear factors, hence 1, 2 or 3 x-intercepts.



A quartic function may have 0, 1, 2, 3 or 4 distinct linear factors, hence 0, 1, 2, 3 or 4 x-intercepts.



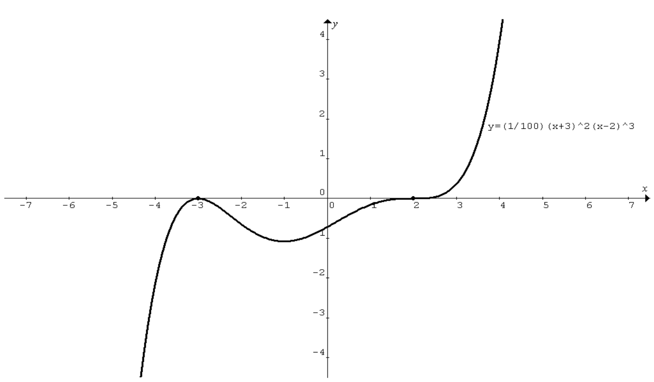
If the power of a linear factor in a polynomial is *even*, then the corresponding x-intercept is a turning point.
If the power of a linear factor in a polynomial is *odd*, then the corresponding x-intercept is a stationary point of inflection.

If $y = a(x-b)^2(x-c)^3(x-d)^4(x^2+e)(x-f)^5$, then the x-intercepts at $x = b$ and $x = d$ are turning points; and the x-intercepts at $x = c$ and $x = f$ are stationary points of inflection. The factor $x^2 + e$ is not linear and $\therefore$ does not correspond to an x-intercept.

If a is a positive (negative) value, the graph of a polynomial function heads upwards (downwards) in the positive x-direction.

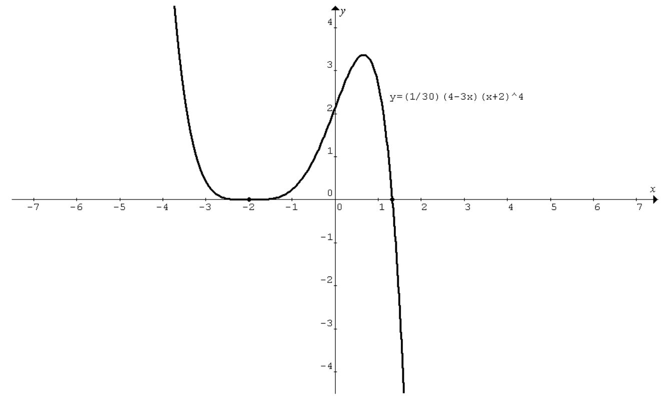
Example 1 Sketch $y = \frac{1}{100}(x+3)^2(x-2)^3$

The x-intercept at $x = -3$ is a turning point; at $x = 2$ the x-intercept is a stationary point of inflection.

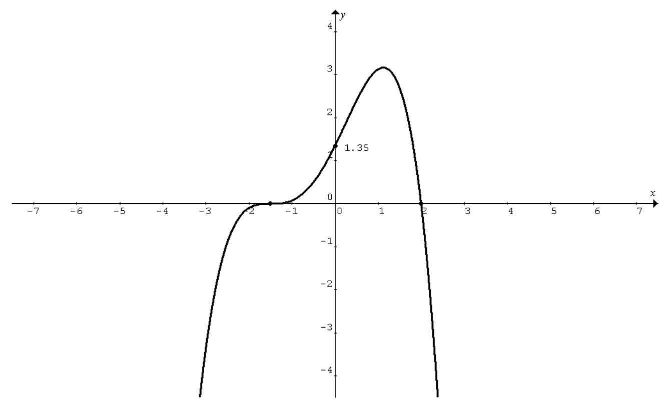


Example 2 Sketch $y = \frac{1}{30}(4-3x)(x+2)^4$

Express the function as $y = -\frac{1}{10}\left(x - \frac{4}{3}\right)(x+2)^4$. The function crosses the x-axis at $x = \frac{4}{3}$; it touches the x-axis at $x = -2$.



Example 3 Find the equation of the quartic function shown below.



At $x = -\frac{3}{2}$ the function has an x-intercept that is a stationary point of inflection; at $x = 2$ the function crosses the x-axis.

Hence, $y = a\left(x + \frac{3}{2}\right)^3(x-2)$, where $|a|$ is the vertical dilation factor to be determined using further information, in this case, the y-intercept $(0, 1.35)$.

$1.35 = a\left(\frac{3}{2}\right)^3(-2)$, $\therefore a = -\frac{1}{5}$

$\therefore y = -\frac{1}{5}\left(x + \frac{3}{2}\right)^3(x-2)$

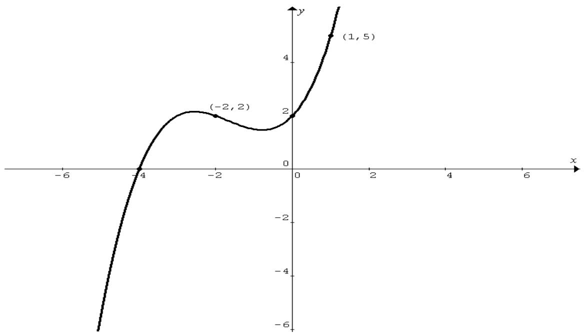
This quartic function can also be expressed as

$y = -\frac{1}{5}\left[\frac{1}{2}(2x+3)\right]^3(x-2) = -\frac{1}{5}\left(\frac{1}{2}\right)^3(2x+3)(x-2)$

$= -\frac{1}{40}(2x+3)(x-2)$

The − sign corresponds to the graph heading 'south' in the positive x-direction.

Example 4 Find the equation of the cubic function shown in the graph below.



The cubic function has only one x-intercept at $x=-4$, and $\therefore$ only one linear factor. Its equation in factorised form must be $y=a(x+4)(x^2+bx+c)$. Use the other given points to set up simultaneous equations, then solve for a, b and c.

$(0,2)\rightarrow 2=4ac \qquad \therefore ac=0.5 \qquad (1)$

$(-2,2)\rightarrow 2=2a(4-2b+c) \qquad \therefore 4a-2ab+ac=1 \qquad (2)$

$(1,5)\rightarrow 5=5a(1+b+c) \qquad \therefore a+ab+ac=1 \qquad (3)$

Substitute eq (1) in eqs (2) and (3)

$4a-2ab=0.5 \qquad (4)$

$a+ab=0.5 \qquad (5) \quad \times 2$

$2a+2ab=1 \qquad (6)$

Add eqs (4) and (6), $6a=1.5$, $\therefore a=0.25 \qquad (7)$

Sub. eq (7) in (5) to obtain $b=1$

Sub. eq (7) in (1) to obtain $c=2$

Hence $y=0.25(x+4)(x^2+x+2)$.

All quadratic functions can be changed to turning point form $y=A(x\pm b)^2\pm c$ by completing the square. The turning point is $(\mp b,\pm c)$.

Some cubic functions can be expressed in similar form $y=A(x\pm b)^3\pm c$. $(\mp b,\pm c)$ is the stationary point of inflection of the cubic function.

Some quartic functions can also be expressed in similar form $y=A(x\pm b)^4\pm c$. $(\mp b,\pm c)$ is the turning point of the quartic function.

These forms should be viewed as the transformations (discussed previously) of the power functions, x^2, x^3 and x^4 respectively.

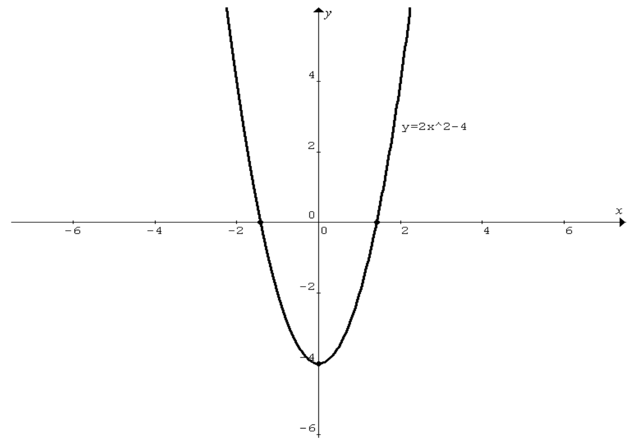
Example 1 Find the turning point and the x-intercepts of $y=2x^2-4$. Sketch its graph.

The function is in turning point form. The turning point is $(0,-4)$.

Factorise $y=2x^2-4=2(x^2-2)=2(x-\sqrt{2})(x+\sqrt{2})$.

The linear factor $x-\sqrt{2}$ gives x-intercept $(\sqrt{2},0)$ and the linear factor $x+\sqrt{2}$ gives x-intercept $(-\sqrt{2},0)$.

The y-intercept is obtained by letting $x=0$, $(0,-4)$.



Example 2 Factorise $2(x+1)^3+2$ and sketch $y=2(x+1)^3+2$.

$y=2(x+1)^3+2$

$=2\left[(x+1)^3+1^3\right]$

$=2[(x+1)+1]\left[(x+1)^2-(x+1)(1)+(1)^2\right]$

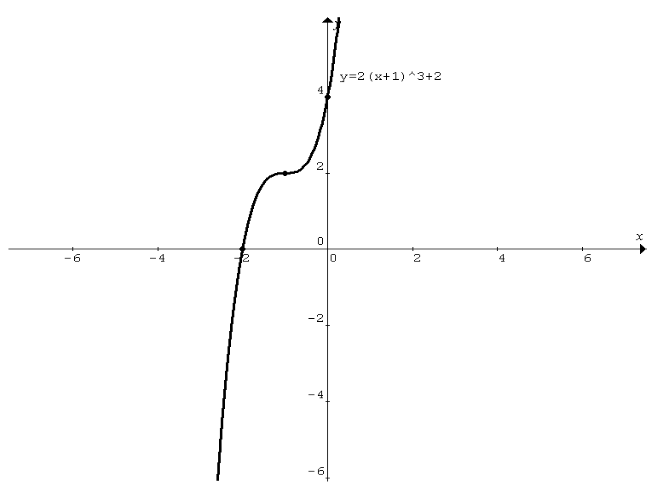
$=2(x+2)(x^2+x+1)$.

There is only one linear factor, $\therefore$ only one x-intercept at $x=-2$.

Note that the x-intercept can also be obtained by letting $y=0$ and solve for x. $2(x+1)^3+2=0$, $2(x+1)^3=-2$, $(x+1)^3=-1$, $x+1=\sqrt[3]{-1}=-1$, $\therefore x=-2$.

Let $x=0$ to obtain $y=4$. y-intercept is $(0,4)$.

The given function is in stationary inflection point form. The stationary point of inflection is $(-1,2)$.



Graphs of sum and difference of functions

The sum (or difference) of two functions f and g is defined only for $x \in D_f \cap D_g$, where D_f and D_g are the domains of f and g respectively.

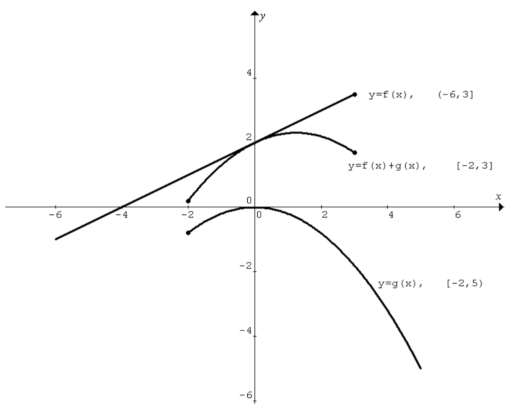
Example Given functions f and g defined by $f(x) = \sqrt{x+1}$ and $g(x) = \log_e(2-x)$ respectively, find the domain of $f + g$.

$f(x) = \sqrt{x+1}$, $\therefore x+1 \geq 0$, $\therefore x \geq -1$, $\therefore D_f = \{x : x \geq -1\}$.

$g(x) = \log_e(2-x)$, $\therefore 2-x > 0$, $\therefore x < 2$, $\therefore D_g = \{x : x < 2\}$.

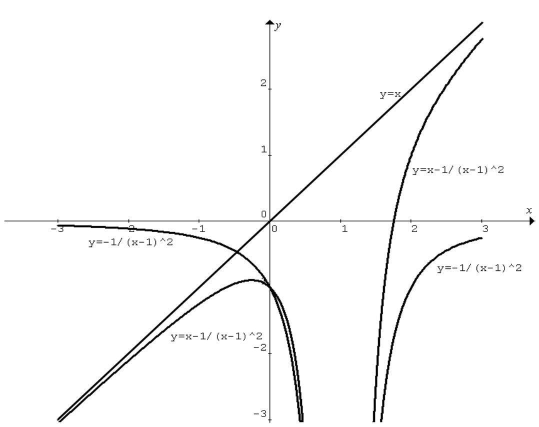
Hence, $D_{f+g} = D_f \cap D_g = \{x : -1 \leq x < 2\}$, or $[-1, 2)$.

If the graphs of $y = f(x)$ and $y = g(x)$ are given, the graph of $y = f(x) + g(x)$ can be sketched by the method of addition of ordinates, i.e. by adding the y-coordinates of the two functions at several suitable x values in $D_f \cap D_g$.



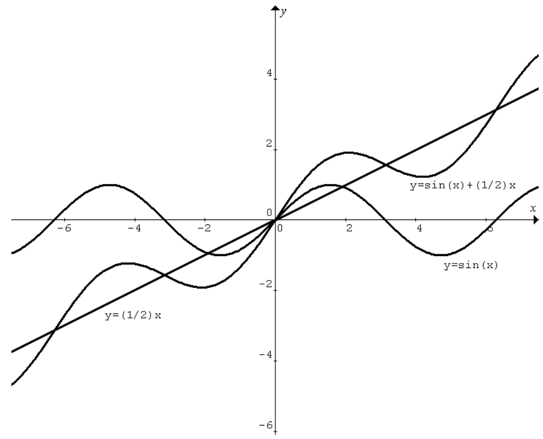
Example 1 Use addition of ordinates to sketch $y = x - \frac{1}{(x-1)^2}$.

Sketch $y = x$ and $y = -\frac{1}{(x-1)^2}$ on the same axes, then add the y-coordinates of the two functions at several suitable x-values. Note that $y = x - \frac{1}{(x-1)^2}$ is undefined at $x = 1$, $\therefore$ its domain is $R \setminus \{1\}$.



Example 2 Sketch $y = \sin x + \frac{1}{2}x$ by addition of ordinates.

Sketch $y = \sin x$ and $y = \frac{1}{2}x$ on the same axes, then add the y-coordinates of the two functions at several suitable x-values.



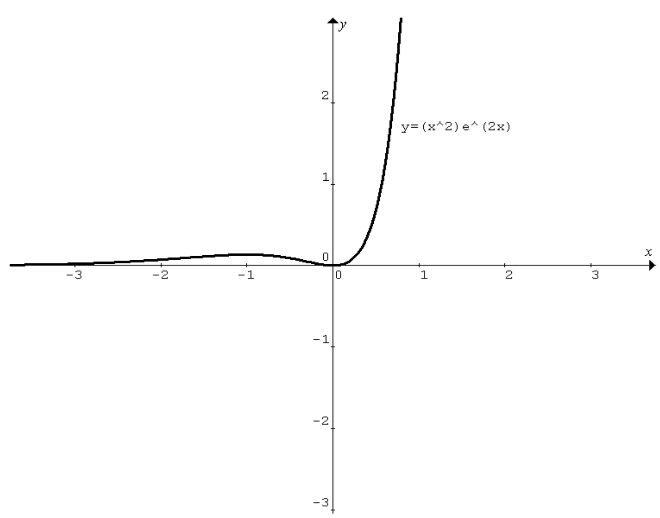
Graph of product of functions

New functions can be generated by addition (or subtraction) of functions as discussed in the previous section. New functions called products (or quotients) of functions can also be generated by multiplication (or division) of functions.

The product (or quotient) of two functions u and v is defined only for $x \in D_u \cap D_v$, and $v \neq 0$ if v is the divisor.

If the graphs of $y = u(x)$ and $y = v(x)$ are given, then the graph of $y = u(x)v(x)$ or $\left(y = \frac{u(x)}{v(x)}\right)$ can be sketched by multiplying (or dividing) the y-coordinate of one function by the y-coordinate of the other at several suitable x values within $D_u \cap D_v$.

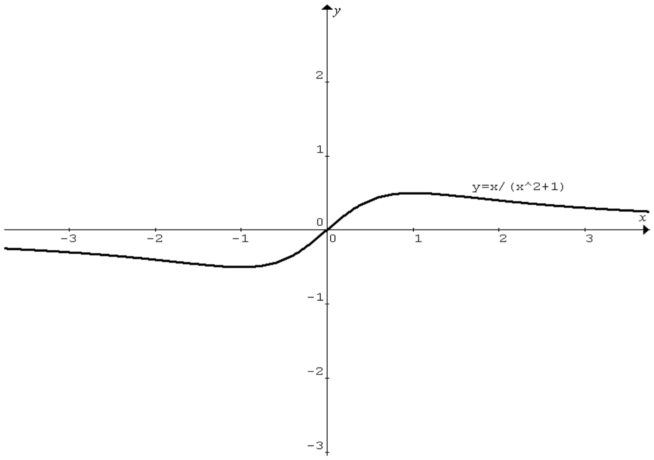
Example 1
$y = x^2 e^{2x}$ is the product of functions $u(x) = x^2$ and $v(x) = e^{2x}$.
$D_u = R$ and $D_v = R$, $D_{uv} = D_u \cap D_v = R$.



Example 2

$y=\dfrac{x}{x^2+1}$ is the quotient of functions u and v defined by $u(x)=x$ and $v(x)=x^2+1$ respectively.

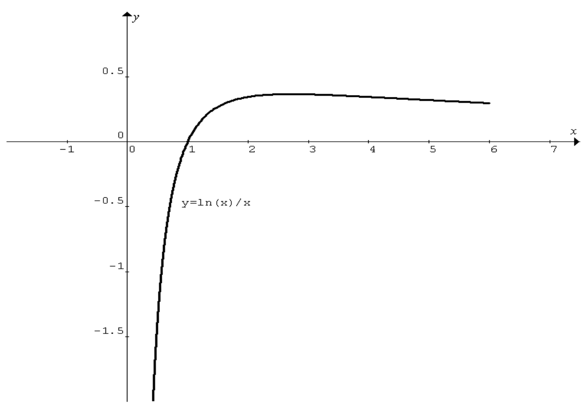
$D_u=R$ and $D_v=R$, $D_{uv}=D_u\cap D_v=R$



Example 3

$y=\dfrac{\log_e x}{x}$ is the quotient of functions u and v defined by $u(x)=\log_e x$ and $v(x)=x$.

$D_u=R^+$ and $D_v=R\setminus\{0\}$, $D_{uv}=D_u\cap D_v=R^+$



Graphs of composite functions

Given two functions *f* and *g* with equations $y=f(x)$ and $y=g(x)$ respectively, new functions can be generated in the following ways:

In $y=f(x)$ the variable *x* is replaced by $g(x)$, $\therefore$ the equation of the new function is $y=f(g(x))$.

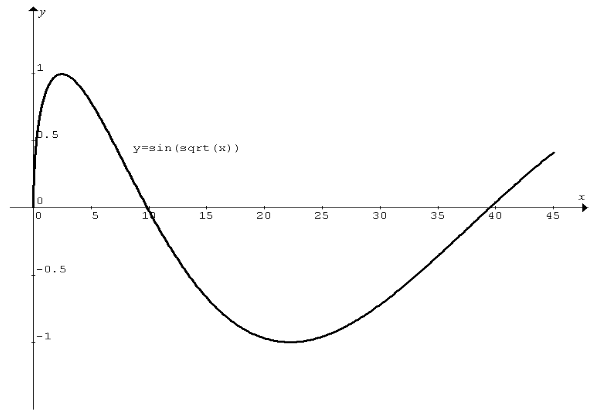
In $y=g(x)$ the variable *x* is replaced by $f(x)$, $\therefore$ the equation of the new function is $y=g(f(x))$.

Functions generated in the above manner are called *composite* functions. The two new composite functions are denoted as $f\circ g$ and $g\circ f$ respectively,

i.e. $f\circ g(x)=f(g(x))$ and $g\circ f(x)=g(f(x))$.

Example 1 Generate two composite functions from functions *f* and *g* defined by $f(x)=\sin x$ and $g(x)=\sqrt{x}$.

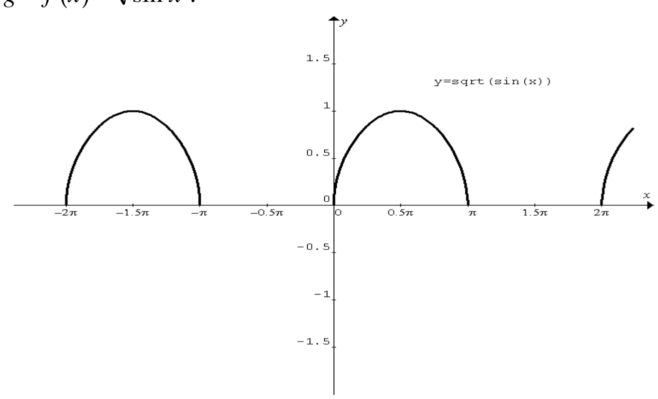
Replace *x* by $\sqrt{x}$ in $f(x)=\sin x$ to obtain $f\circ g(x)=\sin(\sqrt{x})$.



$f\circ g(x)=\sin(\sqrt{x})$ is defined when $x\ge 0$.

Hence $D_{f\circ g}=\{x:x\ge 0\}$.

Replace *x* by $\sin x$ in function $g(x)=\sqrt{x}$ to obtain $g\circ f(x)=\sqrt{\sin x}$.



$g\circ f(x)=\sqrt{\sin x}$ is defined when $\sin x\ge 0$,

i.e. $x\in[2n\pi,(2n+1)\pi]$, where $n=0,\pm1,\pm2,......$

Hence $D_{g\circ f}=\{x:2n\pi\le x\le(2n+1)\pi, n=0,\pm1,\pm2,...\}$

Example 2 Find the domain and range, and sketch the graph of $y=|\cos(2x)|$.

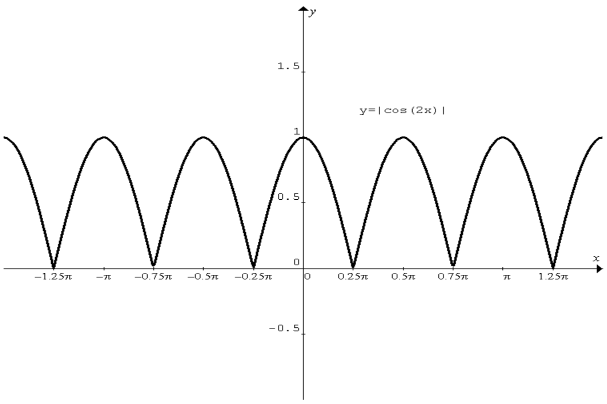
$y=|\cos(2x)|$ is a composite function, $y=f\circ g(x)=f(g(x))$, where $f(x)=|x|$ and $g(x)=\cos(2x)$.

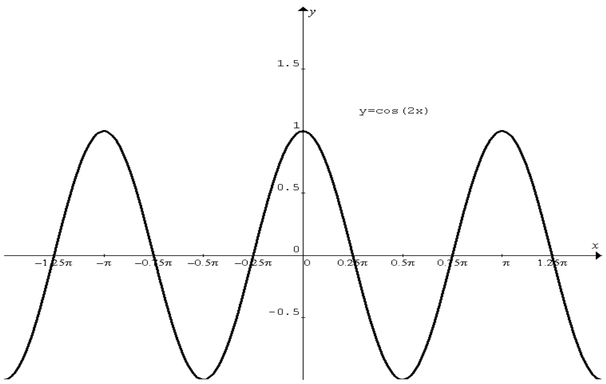
$y=|\cos(2x)|$ is defined for $\cos(2x)\in R$, i.e. $x\in R$.

Hence $D_{f\circ g}=R$.

Since $-1\le\cos(2x)\le 1$, $\therefore 0\le|\cos(2x)|\le 1$.

Hence the range of the composite function is $R_{f\circ g}=[0,1]$.





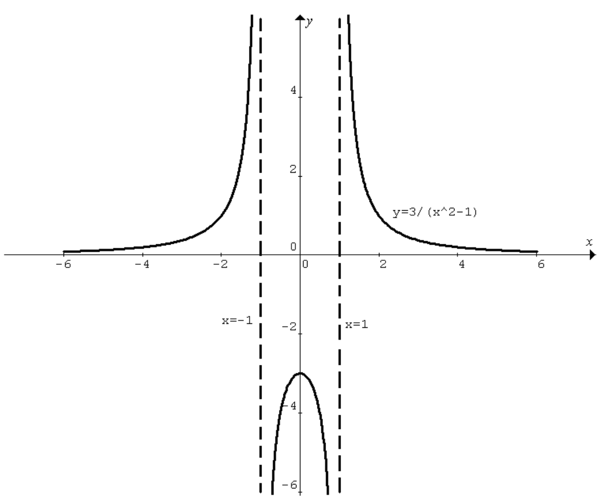
The graph of $y=\cos(2x)$ is also shown for comparison. The negative half is reflected in the *x*-axis.

Example 3   Find the domain and range, and sketch the graph of $y=\dfrac{3}{x^2-1}$.

$y=\dfrac{3}{x^2-1}$ is a composite function, $y=f\circ g(x)=f(g(x))$, where $f(x)=\dfrac{3}{x}$ and $g(x)=x^2-1$.

The function is defined if $x^2-1\neq 0$, i.e. $x\neq\pm1$.
Hence $D_{f\circ g}=R\setminus\{-1,1\}$ and the function has vertical asymptotes $x=-1$ and $x=1$.

The value of the function cannot be zero, $\therefore R_{f\circ g}=R\setminus\{0\}$ and the function has the *x*-axis as a horizontal asymptote $y=0$.

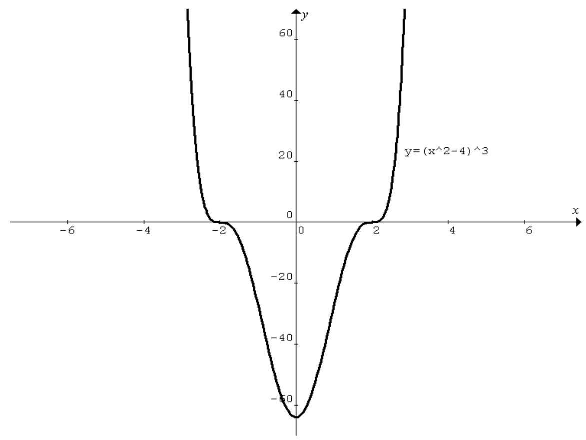


Example 4   Find the domain and range, and sketch the graph of $y=(x^2-4)^3$.

$y=(x^2-4)^3$ is a composite function $y=f\circ g(x)=f(g(x))$, where $f(x)=x^3$ and $g(x)=x^2-4$.

It is defined for all real *x* values.
Hence $D_{f\circ g}=R$.
The lowest value of the function is $(-4)^3=-64$.
Hence $R_{f\circ g}=[-64,\infty)$.

The function can be expressed as the product of the cube of two linear factors, $y=(x^2-4)^3=(x-2)^3(x+2)^3$, $\therefore$ the *x*-intercepts at ±2 are stationary points of inflection.



Graphs of inverse relations

A relation is a set of points. A new set of points can be generated by interchanging the *x* and *y*-coordinates of each point. This new set of points is called the **inverse** of the original relation. The equation of the inverse is obtained by interchanging *x* and *y* in the original equation.

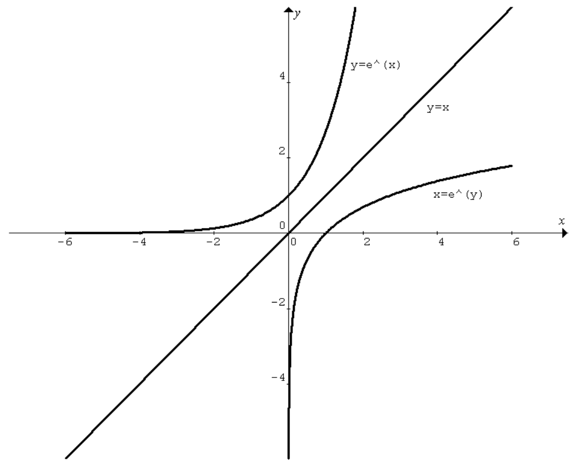
The *y*-intercepts of the original relation become the *x*-intercepts of the new relation, and the *x*-intercepts of the original become the *y*-intercepts of the new relation.

The horizontal asymptotes of the original relation become the vertical asymptotes of the new relation, and the vertical asymptotes of the original become the horizontal asymptotes of the new relation.

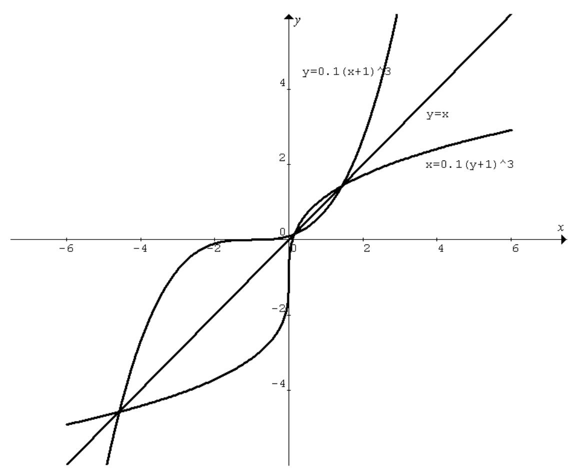
The range of the original relation becomes the domain of the new relation, and the domain of the original becomes the range of the new relation.

Graphically the inverse relation and the original relation are reflections of each other in the line $y=x$.
Note: The same scale for both axes must be used to display this reflection property graphically.

Example 1



Example 2

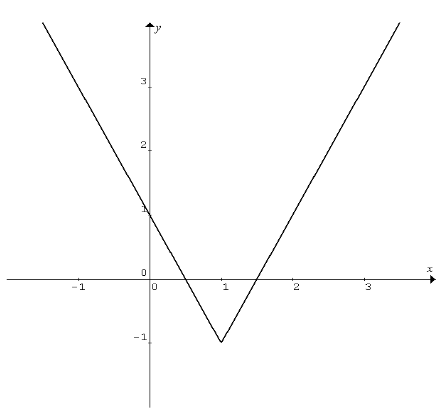


Hybrid functions

In some practical situations a different rule (equation) is required for a different interval in the domain of the function. Such a function is called a hybrid function.

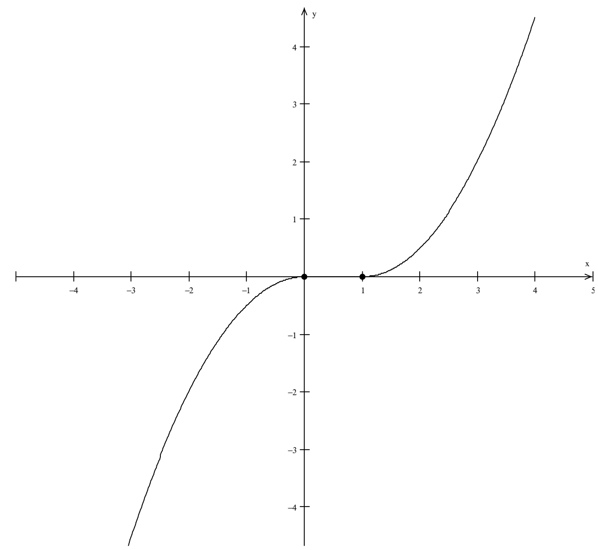
Example 1 The modulus function *f* defined by $f(x)=2|x-1|-1$ can be expressed as a hybrid function

$$f(x)=\begin{cases}2x-3 & for \quad x\ge 1\\ 1-2x & for \quad x<1\end{cases}$$



Example 2 Sketch the graph of $y=f(x)$ where

$$f(x)=\begin{cases}\frac{1}{2}(x-1)^2 & for \quad x>1\\ 0 & for \quad 0<x\le 1\\ -\frac{1}{2}x^2 & for \quad x\le 0\end{cases}$$



Example 3 A paint shop only sells paints in 0.5 L, 1 L, 4 L and 10 L cans. The corresponding prices are \$20, \$30, \$60 and \$100. Draw a graph of the cost per L of paint (\$ *y*) as a function of the amount of paint (*x* L) if you buy just *one* can of paint for a job.

Use a hybrid function to display the given information.

$$y=\begin{cases}40 & for \quad 0<x\le 0.5\\ 30 & for \quad 0.5<x\le 1\\ 15 & for \quad 1<x\le 4\\ 10 & for \quad 4<x\le 10\end{cases}$$