2016 VCAA Specialist Mathematics Sample Exam 1 (v2 April) Solutions

Q1a Let $z = \sqrt{5} - i$,

$(\sqrt{5}-i)^3 - (\sqrt{5}-i)(\sqrt{5}-i)^2 + 4(\sqrt{5}-i) - 4(\sqrt{5}-i) = 0$

.: $\sqrt{5}-i$ is a solution.

Q1b $z^3 - (\sqrt{5}-i)z^2 + 4z - 4(\sqrt{5}-i) = (z-(\sqrt{5}-i))(z^2+4) = 0$

.: $z^2 + 4 = 0$, .: $z = \pm 2i$ are the other solutions.

Q2 $3x^2 + 2xy + y^2 = 11$ and $y > 0$ (in the first quadrant)

At $x = 1$, $-8 + 2y + y^2 = 0$, .: $y = 2$

Implicit differentiation: $\frac{d}{dx}(3x^2 + 2xy + y^2) = 0$,

$6x + 2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{3x+y}{x+y}$

At $(1, 2)$, $\frac{dy}{dx} = -\frac{5}{3}$, .: gradient of the normal $= \frac{3}{5}$

.: equation of the normal: $y - 2 = \frac{3}{5}(x-1)$, $3x - 5y + 7 = 0$

Q3a $\overline{X+Y} = \overline{X} + \overline{Y} = 240 + 10 = 250$ mL

$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) = 8^2 + 2^2 = 68$ (mL)^2

Q3bi Null hypothesis: The second machine is, on average, dispensing **not** less coffee than the first.

Alternative hypothesis: The second machine is, on average, dispensing less coffee than the first.

Q3bii $\text{sd}(\overline{X}) = \frac{8}{\sqrt{16}} = 2$, $a = \frac{235-240}{2} = -2.5$,

$p = \Pr(Z \le -2.5) \approx 0.0062$. Since $p < 0.05$, the null hypothesis should be rejected at the 0.05 level of significance.

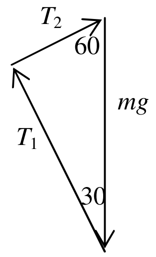
Q4a $V = \int_0^a \pi(e^{-x})^2\,dx = \int_0^a \pi\, e^{-2x}\,dx$

Q4b $V(a) = \pi\left[\frac{e^{-2x}}{-2}\right]_0^a = \pi\left(\frac{1-e^{-2a}}{2}\right)$

Q4c $\pi\left(\frac{1-e^{-2a}}{2}\right) = \frac{5\pi}{18}$, $9 - 9e^{-2a} = 5$, $e^{-2a} = \frac{4}{9}$, $e^{2a} = \frac{9}{4}$,

$e^a = \frac{3}{2}$, $a = \log_e\left(\frac{3}{2}\right)$

Q5a



$\frac{T_2}{T_1} = \tan 30$, .: $T_2 = \frac{T_1}{\sqrt{3}}$

Q5b $\frac{T_2}{mg} = \sin 30$, let $T_2 = 98$

.: $m = \frac{2 \times 98}{9.8} = 20$ is the maximum value.

Q6

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cos^2(2x)\sin(2x)\,dx = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} -\frac{1}{2}u^2\frac{du}{dx}\,dx$

$u = \cos(2x)$

$\frac{du}{dx} = -2\sin(2x)$

$-\frac{1}{2} \times \frac{du}{dx} = \sin(2x)$

$= \int_{-1}^{0} -\frac{1}{2}u^2\,du = \left[-\frac{u^3}{6}\right]_{-1}^{0} = -\frac{1}{6}$

Q7 $\frac{dy}{dx} = \frac{y}{x^2}$, $\int \frac{1}{y}\,dy = \int \frac{1}{x^2}\,dx$, $\log_e|y| = -\frac{1}{x} + c$

Given $x = 1$, $y = -1$, $\log_e|-1| = -\frac{1}{1} + c$, $c = 1$

.: $\log_e|y| = 1 - \frac{1}{x}$, $|y| = e^{\left(1-\frac{1}{x}\right)}$, $y = \pm e^{\left(1-\frac{1}{x}\right)}$

Q8a Arc length

$= \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \int_0^{\pi} \sqrt{(-2\sin(2\theta))^2 + (2\cos(2\theta))^2}\,d\theta$

Q8b Arc length $= 2\int_0^{\pi} \sqrt{\sin^2(2\theta) + \cos^2(2\theta)}\,d\theta = 2\int_0^{\pi} d\theta = 2\pi$

Q9a $\tilde{b} = \tilde{i} + 2\tilde{j} + m\tilde{k}$, $|\tilde{b}| = \sqrt{1^2 + 2^2 + m^2} = 2\sqrt{3}$

.: $m^2 + 5 = 12$, $m = \pm\sqrt{7}$

Q9b $\tilde{a}.\tilde{b} = 0$, $1 - 2 + 2m = 0$, $m = \frac{1}{2}$

Q9ci $3\tilde{c} - \tilde{a} = 2\tilde{i} + 4\tilde{j} - 5\tilde{k}$

Q9cii Since $3\tilde{c} - \tilde{a} = 2\tilde{i} + 4\tilde{j} - 5\tilde{k}$.: $3\tilde{c} - \tilde{a} = 2\tilde{b}$ if $m = -\frac{5}{2}$

.: $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ are linearly dependent if $m = -\frac{5}{2}$

Q10a $\frac{1}{x^2} + \frac{3}{x} + \frac{2x-1}{x^2+4} = \frac{(x^2+4) + 3x(x^2+4) + x^2(2x-1)}{x^2(x^2+4)}$

$= \frac{5x^3 + 12x + 4}{x^2(x^2+4)}$

Q10b $\int \frac{5x^3 + 12x + 4}{x^2(x^2+4)}\,dx = \int \frac{1}{x^2} + \frac{3}{x} + \frac{2x-1}{x^2+4}\,dx$

$= \int \frac{1}{x^2} + \frac{3}{x} + \frac{2x}{x^2+4} - \frac{1}{x^2+4}\,dx$

$= -\frac{1}{x} + 3\log_e|x| + \log_e(x^2+4) - \tan^{-1}\left(\frac{x}{2}\right)$

$= -\frac{1}{x} + \log_e|x^3|(x^2+4) - \tan^{-1}\left(\frac{x}{2}\right)$

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