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2016 Specialist Mathematics

Year 12 Application Task

Time allowed: 5 hours

You are allowed: 1 bounded reference, 1 CAS, 1 scientific calculator
Working must be shown for questions worth 2 or more marks. Total 190 marks

Theme: Hiking near a developing country town

Position in the Cartesian plane can be represented by a complex number in the Argand diagram.

A country town centre is chosen as the origin O of the Cartesian plane and the Argand diagram.

1 unit in the real part $\mathrm{Re}(z)$ or imaginary part $\mathrm{Im}(z)$ of a complex number z represents 1 km.

The $\mathrm{Re}(z)$ axis points to the east, and the $\mathrm{Im}(z)$ axis points to the north.

Measure directions in °T ***(true bearings)***

Campsite C is 2 km east and 1.5 km south of the town centre.

Question 1

a. Let complex number z_C represent the position of the campsite. Find z_C in $x + i\,y$ form. 1 mark

b. Find z_C in exact polar form. 2 marks

c. Find the straight-line distance of Campsite C from the town centre. 1 mark

d. Find the direction (true bearing, correct to 1 decimal place) of Campsite C from the town centre. 1 mark

e. Hiker A starts from Campsite C travelling $\sqrt{2}$ km in a straight line in the direction $45°\text{T}$. Let complex number z_{A} represent the position of Hiker A from the town centre at the end of his travel.
Find z_{A} in $x+i\,y$ form. 2 marks

f. Hiker B starts from Campsite C travelling ℓ km in a straight line in the direction $\theta°\text{T}$, where $10<\theta\leq 90$. Let complex number z_{B} represent the position of Hiker B from the town centre at the end of his travel. Find z_{B} in terms of ℓ and θ.

3 marks

g. Find $\text{Re}(z_{\text{B}})$ and $\text{Im}(z_{\text{B}})$ in terms of ℓ and θ. Express your answers in simplest form. 2 marks

h. Write an expression for the distance d_{AB} separating Hiker A and Hiker B in terms of z_{A} and z_{B}. 2 marks

i. Write an expression for the distance d_{AB} separating Hiker A and Hiker B in terms of ℓ and θ. 3 marks

j. Show that $d_{\text{AB}}{}^2 = 6 - 4(\sin\theta + \cos\theta)$ when $\ell = 2$. 2 marks

k. With the help of CAS, find the exact maximum and minimum distances separating Hiker A and Hiker B when $\ell = 2$.

4 marks

The country town chief places 6 markers around the town.
The markers are spaced out equally in a circle of radius 1 km. The town centre is at the centre of the circle. One of the 6 markers is to the west of the town centre.

l. Let z be the position of any one of the 6 markers. Write a polynomial equation in z satisfied by the 6 markers, i.e. an equation when solved will give the positions of the 6 markers.

1 mark

m. Write down the exact position in polar form of each of the 6 markers. 3 marks

n. Write down the exact position in $x + i\,y$ form of each of the 6 markers. 3 marks

The campsite supervisor also places 6 markers around the campsite.
The markers are spaced out equally in a circle of radius 1 km. The campsite is at the centre of the circle. One of the markers is to the west of the campsite.

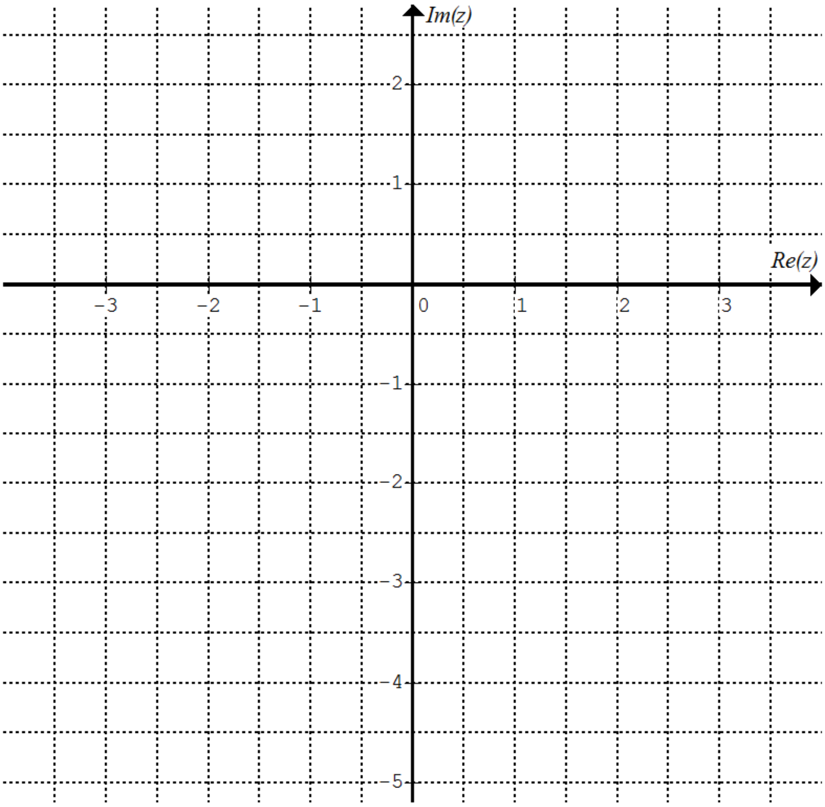
o. Let z be the position of any one of the 6 markers. Write a polynomial equation in z satisfied by the 6 markers.

1 mark

p. Write down the exact position in $x + i\,y$ form of each of the 6 markers. 3 marks

q. Plot accurately the positions of the 6 markers around Campsite C in the following Argand diagram.

3 marks



r. The positions of the 6 markers around Campsite C are labelled in order as z_1, z_2, z_3, z_4, z_5 and z_6. Find the angle between the line segment joining z_1 to z_5 and the line segment joining z_1 to z_6.

2 marks

Now Hiker A is at $(1,-1)$ and sends a laser ray in the direction $45°\text{T}$. Hiker B is at $(\beta,0)$, where $\beta > 5$, and sends a laser ray in the direction $330°\text{T}$.

Note: Do not confuse the set of complex numbers C with Campsite C.

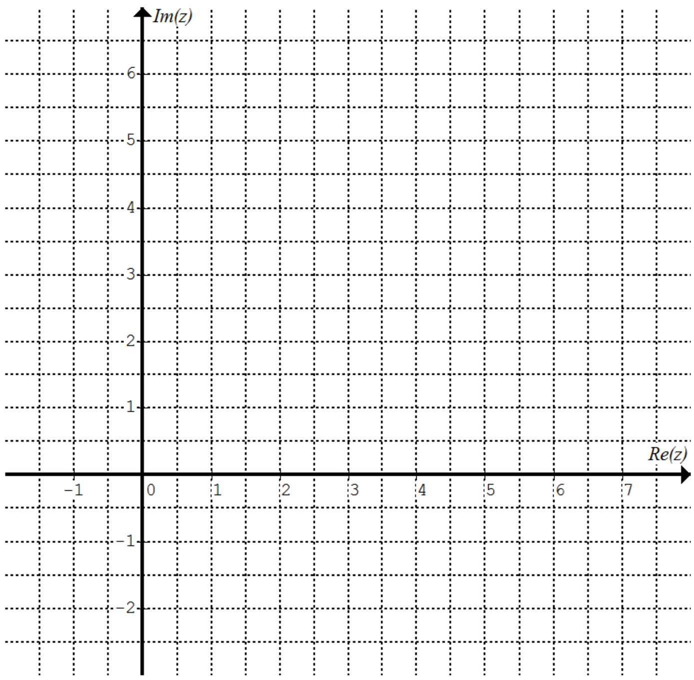
Question 2

a. Express the path of the laser ray sent by Hiker A as a subset of C, the set of complex numbers. 2 marks

b. Express the path of the laser ray sent by Hiker B as a subset of C in terms of β. 2 marks

c. Sketch accurately the two subsets of C in the following diagram. Label the position of Hiker A as w_A, and the position of Hiker B as w_B.

4 marks



d. Let $\phi°$ be the acute angle between the two laser rays. Determine the value of ϕ. 3 marks

e. Determine the exact value of $\tan\phi°$. 4 marks

f. The two laser rays intersect at position represented by $w_{\mathrm{I}} = 4 + 2i$. Find the exact value of β in $(\beta, 0)$, i.e. w_{B}, the position of Hiker B.

3 marks

A third hiker, Hiker P, is at position $w_{\mathrm{P}} = h + ki$.

g. Hiker P sends a third laser ray. This ray passes through $w_{\mathrm{I}} = 4 + 2i$ and $w_{\mathrm{J}} = 4$. Determine the possible values of h and k.

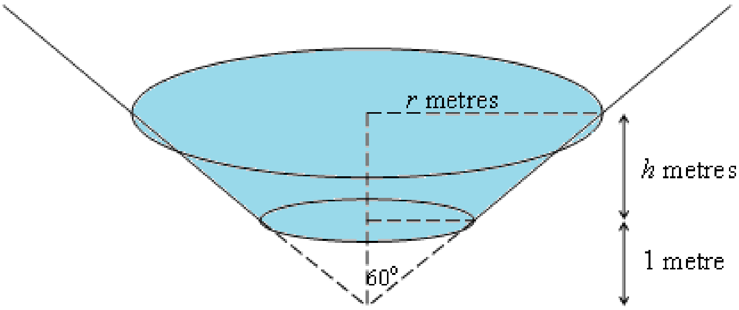
4 marks

h. Find w_{P} if Hiker P is **equidistant** from Hiker A and Hiker B. 4 marks

i. Find w_{P} if the line segment joining Hiker P to Hiker A is **perpendicular** to the line segment joining Hiker P to Hiker B.

5 marks

The country town chief wants to build a circular pool of water in the shape of a truncated inverted cone as shown in the following diagram. The radius of the water surface is r metres when the water is h metres deep.



Question 3

a. Show that $r = \sqrt{3}(h+1)$. 1 mark

b. Show that the radius of the bottom of the pool is $\sqrt{3}$ metres. 1 mark

c. Show that the area A (m^2) of the water surface is $3\pi(h+1)^2$ when the water is h metres deep. 1 mark

d. Show that the volume V (m^3) of water in the pool is $\pi(h^3 + 2h^2 + h - 1)$ when the water is h metres deep.

3 marks

e. Show that $V=\frac{1}{\sqrt{\pi}}\left(\frac{A}{3}\right)^{\frac{3}{2}}-\frac{A}{3}-\pi$. 3 marks

Water runs into the pool at 10000 litres per hour. (Note: the volume of 1000 litres of water is 1 m^3)

f. Find the exact value of $\frac{dV}{dt}$ in m^3 per min. 2 marks

g. Find the exact value of $\frac{dh}{dt}$ in m per min when the depth of water is 1 m. 3 marks

h. Find the exact value of $\dfrac{dr}{dt}$ in m per min when the depth of water is 1 m. 2 marks

i. Find the exact value of $\dfrac{dA}{dt}$ in m^2 per min when the depth of water is 1 m. 3 marks

j. Find the exact value of $\dfrac{dA}{dV}$ when the depth of water is 1 m. State the unit in your answer. 3 marks

The pool is filled to capacity when the water surface is $12\pi\ \mathrm{m}^2$ and is levelled with the ground.
The pool is empty initially.

k. Find the exact time (min) taken to fill the pool. 2 marks

The centre of the pool is to be at the midpoint between the country town centre and Campsite C.

l. Write down the location of the midpoint in $x+iy$ form. (Distance in km) 1 mark

m. Write down the set of complex numbers representing the water **surface** when the pool is filled to capacity. Be careful with different length units.

3 marks

To attract more tourists, the country town chief has a second thought.
The chief wants to build an elliptical-shape artificial lake instead of the circular pool.
If the centre of the lake is 1 km north of the town centre, the perimeter of the lake can be represented by the set of complex numbers $z=\cos\lambda+\left(1+\frac{1}{2}\sin\lambda\right)i$, where $\lambda\in[0, 2\pi)$ is a parameter.
Distances are measured in km.

Question 4

a. Find the Cartesian equation of the perimeter of the elliptical lake if it is to be built with the lake centre at 1 km north of the town centre.

3 marks

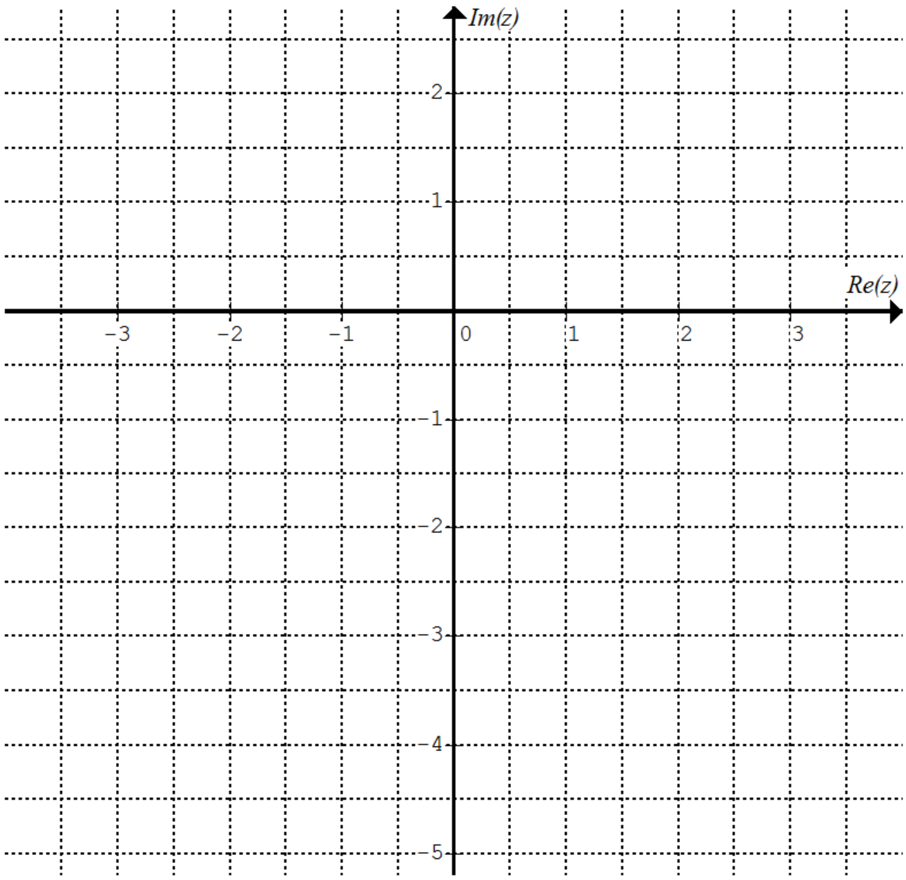
b. Find the maximum distance $d_{\max}$ from one side of the lake to the other side through the centre of the lake.

1 mark

c. Find the minimum distance $d_{\min}$ from one side of the lake to the other side through the centre of the lake.

1 mark

d. Sketch the lake in the following diagram. 2 marks



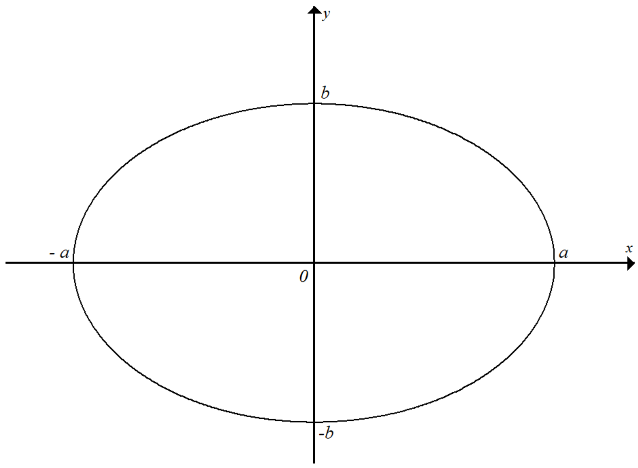
e. The area of the lake is given by $A = \dfrac{\pi\, d_{\min} d_{\max}}{4}$.

Calculate the exact area of the lake in km^2. 1 mark

f. Write down a definite integral for the surface area of the lake. Use CAS to find the anti-derivative of the integrand. Evaluate the exact area of the lake. Show working.

6 marks

g. Find the Cartesian equation of the ellipse shown below. Use calculus to show that $A = \pi ab$ for the region enclosed by the ellipse. Hence show that $A = \dfrac{\pi \, d_{\min} d_{\max}}{4}$ for the artificial lake. 6 marks



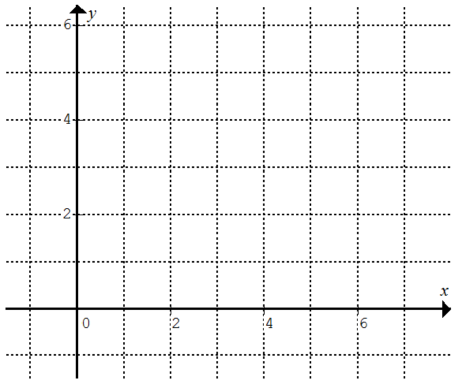
h. The perimeter of the elliptical lake can also be represented by $|z-p-i|+|z+p-i|=q$, where $z \in C$ and $p, q \in R^+$. Determine the exact value of p and q.

12 marks

The country town chief also plans to build a walking track from the town centre $(0, 0)$. The walking track has the equation $y = \sqrt{x}$. It finishes at $(4, 2)$. All length measures are in km.

Question 5

a. Sketch the graph of the walking track. 2 marks



b. Write down a definite integral for the length of the walking track.
Express the integrand as a function of x. Use CAS to evaluate the length in km, correct to 3 decimal places.

4 marks

c. Find the coordinates (correct to 3 decimal places) of a hiker after walking 1.5 km from the town centre. (Use CAS to evaluate)

3 marks

d. Find the true bearing of the hiker from the country town centre. 1 mark

Consider the artificial elliptical lake discussed in Part c.

Question 6

a. Write down a definite integral in terms of parameter λ for the perimeter (km, correct to 3 decimal places) of the artificial elliptical lake. Use CAS to evaluate the definite integral.

6 marks

The walking track will reach the water edge of the lake, and a level bridge over the water is required for the continuation of the track. The bridge has the same equation $y = \sqrt{x}$ as the walking track.

b. Calculate the length (km, correct to 3 decimal places) of the bridge. (Use CAS to evaluate) 4 marks

c. Calculate the area (km^2, correct to 3 decimal places) of the lake on the south side of the bridge. (Use CAS to evaluate) 3 marks

The final decision of the country chief is to build a circular artificial lake and a walking track. The lake centre M is to be 2 km east of the country town centre. The walking track is of the form $y = -k\sqrt{x}$, $k > 0$.
The walking track starts from the town centre towards Campsite C.

Question 7

a. Write down the Cartesian equation of the circular lake of radius r km. 1 mark

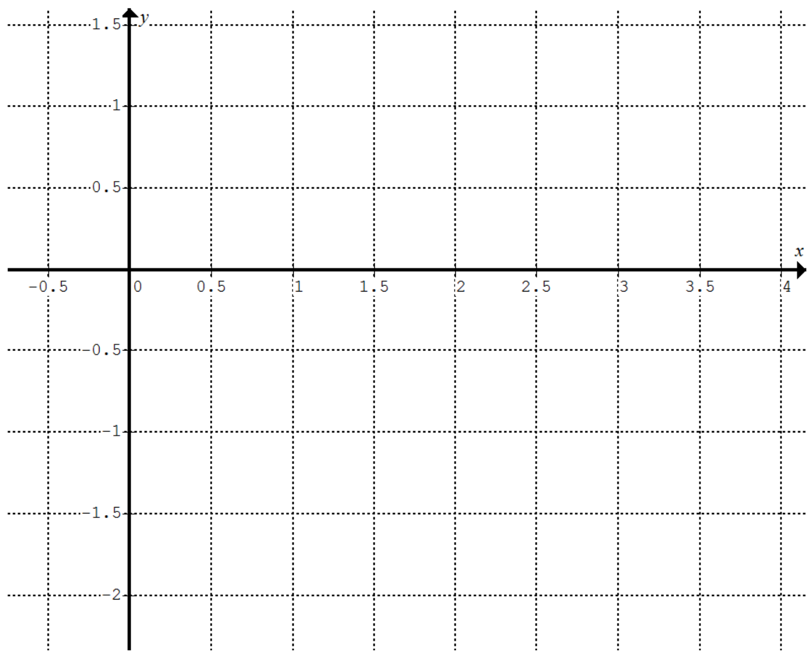
The walking track meets the edge of the circular lake at one point only. This point is labelled as N.

b. Find r in terms of k. 6 marks

c. The walking track finishes at Campsite C. Find the exact value of k . 2 marks

d. Hence find the exact value of r , the radius of the circular lake. 1 mark

e. Sketch the graphs of the circular lake and the walking track. 3 marks



There will be boats for hire at point N.
Consider a person wanting to reach the extreme eastern edge, point E, of the lake.
X is a point at the lake edge between point N and point E.
The person can row in a straight line from N to X at 2.5 km per h and walk δ km from X to E along the edge of the circular lake at 5.0 km per h.

f. Mark down points M, N, X and E on the graph above. Mark the route from N to E taken by the person. Show the coordinates of point N (correct to 3 decimal places)

2 marks

g. Show that $\angle \mathrm{NME} \approx 1.986$ radians. 2 marks

h. Let $\varphi = \angle \mathrm{NMX}$. Express $\angle \mathrm{XME}$ in terms of φ. Hence show that $\delta \approx 2.762 - 1.391\varphi$. 2 marks

i. Show that the length of straight line NX $= \dfrac{495\sin\varphi}{512}$. 2 marks

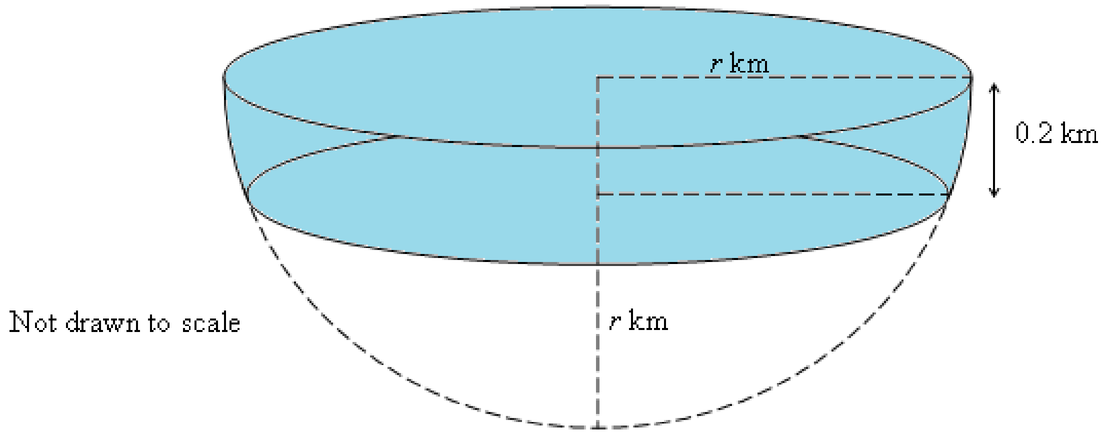
j. Let T (in hours) be the total time taken by the person travelling from point N to point E via point X. Show that $T \approx 0.387\sin\varphi - 0.278\varphi + 0.552$.

3 marks

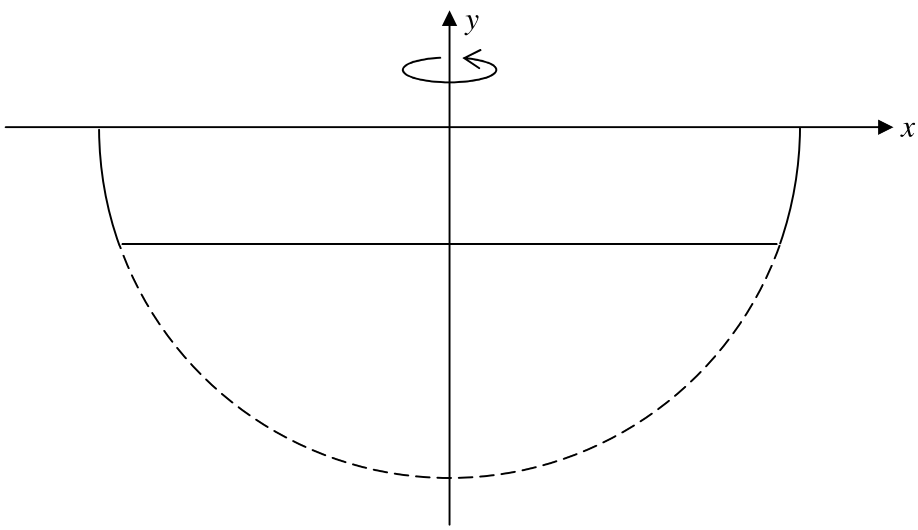
k. Find δ to minimise the travelling time for the journey form point N to point E. 3 marks

l. Find the minimum time of travel for the journey from point N to point E. 2 marks

The circular lake is a truncated hemisphere as shown in the following diagram.
The lake has a depth of 0.2 km.



The semi-circle in the above diagram is redrawn with axes added in the following graph. The truncated hemisphere (i.e. the water in the lake) is the solid of revolution of part of the semi-circle about the *y*-axis.



Question 8

a. Write down the equation and domain of the semi-circle. Express y in terms of x. 2 marks

b. Neatly shade the region to be rotated about the *y*-axis to form a solid of revolution. 1 mark

c. In terms of r, find a definite integral for the volume of the solid of revolution. 2 marks

d. Use the value of r found in **Question 7 d**, evaluate (correct to 3 decimal places) the definite integral in km^3.

2 marks

e. What is the volume of water (in kilolitres, correct to 3 decimal places) in the circular lake? 1 mark

End of task

