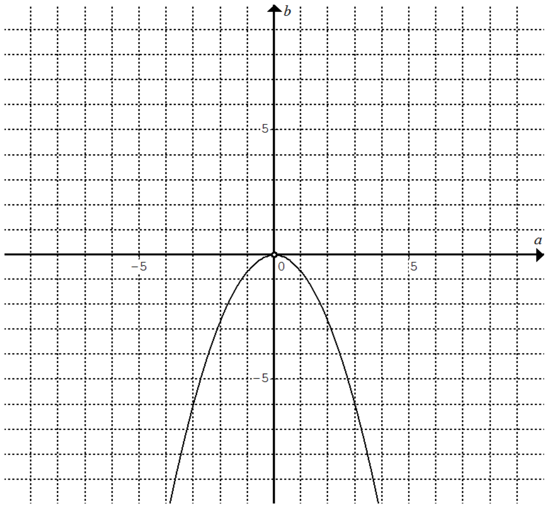
2015 Mathematical Methods (CAS) Trial Exam 1 Solutions

Q1a Let $2(x-a)^2+b=-x^2$, expand and collect like terms,
$3x^2-4ax+\left(2a^2+b\right)=0$
To have one point of contact, $\Delta=0$, .: $2a^2+3b=0$
Pick a value for a, say $a=3$, then $b=-6$

Q1b $b=-\dfrac{2}{3}a^2$



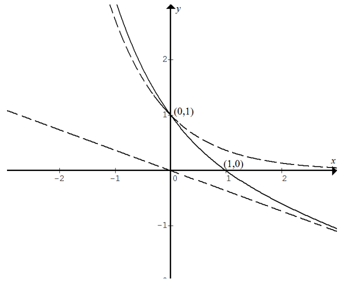
Q2a $y=\dfrac{1}{x+1}+1 \rightarrow x=\dfrac{1}{y+1}+1 \rightarrow x+2=\dfrac{1}{y+1}+1$

$\rightarrow x+2=\dfrac{1}{y-2+1}+1$, simplify and write y as the subject of the equation, $y=\dfrac{1}{x+1}+1$

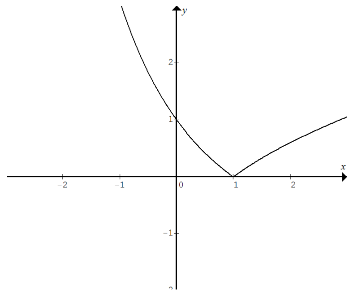
Q2b $\dfrac{1}{x+1}+1\ge 0$, $x\ne -1$ and $\dfrac{1}{x+1}\ge -1$
If $x+1>0$, $x>-1$ and $1\ge -x-1$, i.e. $x\ge -2$, .: $x>-1$
If $x+1<0$, $x<-1$ and $1\le -x-1$, i.e. $x\le -2$, .: $x\le -2$
.: D is $(-\infty,-2]\cup(-1,\infty)$

Q2c $\left(a,\dfrac{5\pi}{6}\right)$ is a continuous interval,
.: the range of g is also a continuous interval
$h\circ g$ is defined if the range of $g\subseteq D$
.: the range of $g\subseteq(-1,\infty)$
.: $2\sin a=-1$, $a=-\dfrac{\pi}{6}$

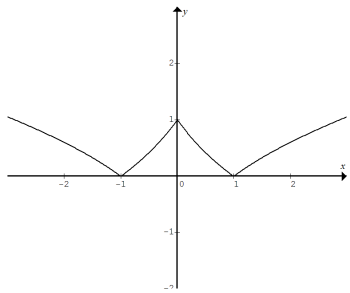
Q3a



Q3b



Q3c



Q4a Let $\sqrt{2x+2}-2=\sqrt{x}-1$, $\sqrt{2x+2}=\sqrt{x}+1$ where $x\ge 0$ and $2x+2>0$, i.e. $x\ge 0$
$\left(\sqrt{2x+2}\right)^2=\left(\sqrt{x}+1\right)^2$, $2x+2=x+2\sqrt{x}+1$, $x+1=2\sqrt{x}$
$(x+1)^2=\left(2\sqrt{x}\right)^2$, $x^2+2x+1=4x$, $x^2-2x+1=0$, $(x-1)^2=0$
.: $x=1$ and $y=0$, the intersection is $(1,0)$.

Q4b $y=\sqrt{2x+2}-2$, $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{2x+2}}$

$y=\sqrt{x}-1$, $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x}}$

The gradient of the common tangent is $-\dfrac{b}{a}$.

.: $-\dfrac{b}{a}=\dfrac{1}{\sqrt{2x+2}}=\dfrac{1}{2\sqrt{x}}$, .: $2x+2=4x$, $x=1$ and $y=0$

.: the common tangent $\dfrac{x}{a}+\dfrac{y}{b}=1$ is at $(1,0)$ and has a gradient of $-\dfrac{b}{a}=\dfrac{1}{2}$

.: $\dfrac{1}{a}+\dfrac{0}{b}=1$, $a=1$ and $b=-\dfrac{1}{2}$

Q5a $2x-1>0$ and $x+1>0$, .: $x>\frac{1}{2}$ and $x>-1$, .: $x>\frac{1}{2}$

The domain is $\left(\frac{1}{2},\infty\right)$.

Q5b As $x\to 0.5^{+}$, the value of $f(x)\to-\infty$, .: $x=\frac{1}{2}$ is an asymptote of $y=f(x)$. It is the only one.

Q5c Let $2\log_{10}(2x-1)-\log_{10}(x+1)=0$.

.: $\log_{10}\frac{(2x-1)^2}{x+1}=0$, .: $\frac{(2x-1)^2}{x+1}=1$

Expand and simplify to $4x^2-5x=0$, $x(4x-5)=0$

Since $x>\frac{1}{2}$, .: $x=\frac{5}{4}$ and $y=0$.

The only x-intercept is $\left(\frac{5}{4},0\right)$.

Q6 $\sin 46°=\sin(45°+1°)=\sin\left(\frac{\pi}{4}+\frac{\pi}{180}\right)$

$\approx\sin\frac{\pi}{4}+\frac{\pi}{180}\times\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{\pi}{180}\times\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\left(1+\frac{\pi}{180}\right)$

Q7a $\frac{dy}{dx}=e^x(\cos x+\sin x)+e^x(\sin x-\cos x)=2e^x\sin x$

Q7b $\frac{dy}{dx}=2e^x\sin x$, .: $\int_0^{\frac{\pi}{3}}2e^x\sin x\,dx=\left[e^x(\sin x-\cos x)\right]_0^{\frac{\pi}{3}}$

.: $\int_0^{\frac{\pi}{3}}e^x\sin x\,dx=\frac{1}{2}\left[e^x(\sin x-\cos x)\right]_0^{\frac{\pi}{3}}$

$=\frac{1}{2}\left(e^{\frac{\pi}{3}}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)-e^0(0-1)\right)=\frac{1}{4}e^{\frac{\pi}{3}}\left(\sqrt{3}-1\right)+\frac{1}{2}$

Q8a Equation of the inverse: $(y-1)^2+1=x$, $(y-1)^2=x-1$, $y=1\pm\sqrt{x-1}$

Q8b It is the same area as the region bounded by $y=(x-1)^2+1$ and $y=2$. When $y=2$, $2=(x-1)^2+1$, $x=0,2$

Area $=\int_0^2\left(2-\left[(x-1)^2+1\right]\right)dx=\int_0^2\left(1-(x-1)^2\right)dx$

$=\left[x-\frac{(x-1)^3}{3}\right]_0^2=\frac{4}{3}$

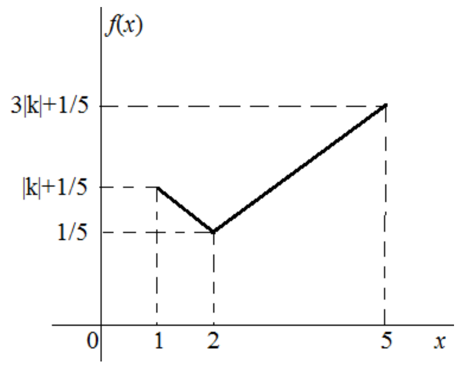
Q9 Binomial distribution, $N=5$, $p=\frac{1}{2}$

x	0	1	2	3	4	5
$\Pr(X=x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

$\Pr(X\ge n)=\frac{13}{16}$, .: $n=2$

$\Pr(X\le 2)=\frac{1}{32}+\frac{5}{32}+\frac{10}{32}=\frac{1}{2}$

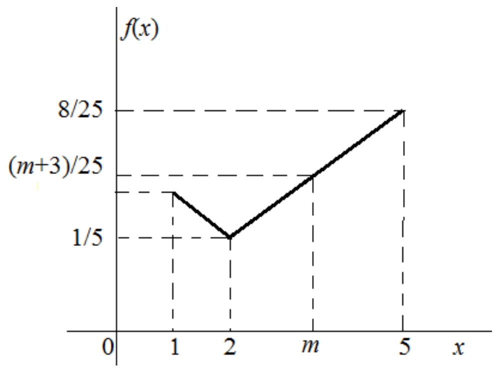
Q10a $f(1)=k+\frac{1}{5}$, $f(2)=\frac{1}{5}$, $f(5)=3k+\frac{1}{5}$



Area under graph $=\frac{1}{2}\left(|k|+\frac{1}{5}+\frac{1}{5}\right)+\frac{3}{2}\left(3|k|+\frac{1}{5}+\frac{1}{5}\right)=1$

$\frac{10|k|}{2}+\frac{4}{5}=1$, $|k|=\frac{1}{25}$

Q10b By inspection of the graph, the median $m\in[2,5]$



$f(x)=\frac{1}{25}(x-2)+\frac{1}{5}=\frac{x+3}{25}$

.: $f(m)=\frac{m+3}{25}$

Area under the graph from $x=m$ to $x=5$:

$\frac{1}{2}\left(\frac{m+3}{25}+\frac{8}{25}\right)(5-m)=\frac{1}{2}$, $m^2+6m-30=0$ and $m>0$

.: $m=-3+\sqrt{39}$

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