2014 VCAA Math Methods CAS Exam 2 Solutions

CAS should be used whenever possible to speed up the solution process.

SECTION 1

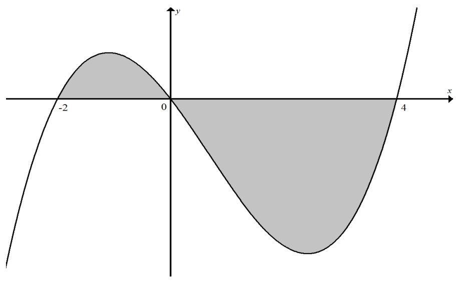
1	2	3	4	5	6	7	8	9	10	11
A	D	D	E	C	B	E	A	A	E	E

12	13	14	15	16	17	18	19	20	21	22
C	C	E	B	C	B	B	C	D	D	E

Q1 $(4,-3)\rightarrow(4,1)\rightarrow(-4,1)$ A

Q2 $y=4-x$
When $y=-2$, $-2=4-x$, $x=6$
When $y=6$, $6=4-x$, $x=-2$
The domain is $(-2,6]$ D

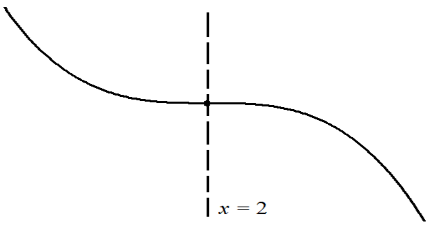
Q3



The required region is shaded.

$A=\int_{-2}^{0}\left(x^3-2x^2-8x\right)dx+\left|\int_{0}^{4}\left(x^3-2x^2-8x\right)dx\right|=\frac{148}{3}$ D

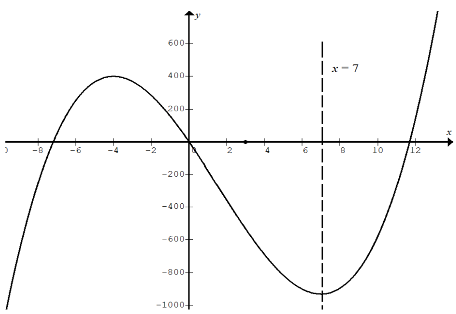
Q4



E

Q5 $\Pr(X<11.5)=\Pr(z<-1)=\Pr(z>1)$ C

Q6

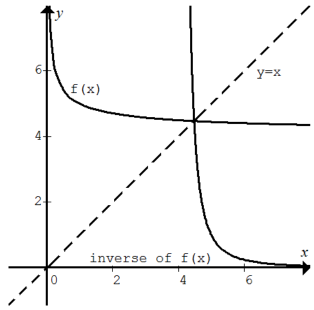


f is a one-to-one function in the interval $(7,\infty)$. B

Q7 $y=a|x+2|+3$ and $(0,0)$, .: $a=-\frac{3}{2}$ E

Q8 $\int_1^4(5-2f(x))dx=[5x]_1^4-2\int_1^4 f(x)dx=15-12=3$ A

Q9



Domain of the inverse is $(4,\infty)$ A

Q10 $f(f(x))=f(2-x)=2-(2-x)=x$ E

Q11 Pr(different colours)
$=\Pr(\text{first red and second blue})+\Pr(\text{first blue and second red})$
$=\frac{5}{9}\times\frac{4}{8}+\frac{4}{9}\times\frac{5}{8}=\frac{5}{9}$ E

Q12 $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}-x+1\\2y-2\end{bmatrix}=\begin{bmatrix}x'\\y'\end{bmatrix}$, .: $x=1-x'$ and $y=\frac{y'+2}{2}$
.: $x-2y=3\rightarrow -x'-y'=4$ C

Q13 $h(x)=\cos(\log_a(x))$ is a one-to-one function and $a>1$.
Possibly, $0\le\log_a(x)\le\pi$.
$\log_a(x)$ is an increasing function for $a>1$.
.: $a^0\le x\le a^\pi$, .: $1\le x\le a^\pi$ C

Q14 $\Pr(X<5\,|\,X<8)=\frac{\Pr(X<5\cap X<8)}{\Pr(X<8)}$
$=\frac{\Pr(X<5)}{\Pr(X<8)}=\frac{1-a}{1-b}=\frac{a-1}{b-1}$ E

Q15 $V=x(6-2x)(8-2x)$, V is a maximum when $x\approx1.1$ B

Q16 $E(X^2)=\int_{-\infty}^{\infty}x^2p(x)dx$, $Var(X)=E(X^2)-\mu^2$
.: $5=\int_{-\infty}^{\infty}x^2p(x)dx-2^2$, .: $\int_{-\infty}^{\infty}x^2p(x)dx=9$ C

Q17 $y=\frac{a}{3}x-\frac{5}{3}$ and $y=\frac{3}{a}x-\left(\frac{8}{a}-1\right)$
To have no solution (intersection), the gradient must be the same and the y-intercept must be different, i.e. $\frac{a}{3}=\frac{3}{a}$ and $\frac{8}{a}-1\ne\frac{5}{3}$,
.: $a=-3$ B

Q18 $y = x^2 + 2x$ and $y = kx - 4$

.: $x^2 + 2x = kx - 4$, $x^2 + (2-k)x + 4 = 0$

For 2 intersections, $\Delta > 0$, $(2-k)^2 - 4(1)(4) > 0$

.: $k < -2$ or $k > 6$ **B**

Q19 $y = 1$, $x = 1$; $y = 2$, $x = 4$; $y = 3$, $x = 9$; $y = 4$, $x = 16$

Jake's estimated area: $1\times1 + 4\times1 + 9\times1 + 16\times1 = 30$

Anita's exact value by finding the area under the inverse:

$\int_0^4 x^2 dx = \frac{64}{3}$

The difference: $30 - \frac{64}{3} = \frac{26}{3}$ **C**

Q20 The average value of $h = \frac{\text{area under } h(x) \text{ from 1 to 11}}{11-1}$

$= \frac{2\left(\frac{1}{2}(4+10)5\right)}{10} = 7$ **D**

Q21 Top edge $= p + 2p\cos x$; height $= p\sin x$ where $0 < x < \frac{\pi}{2}$

Area: $A = \frac{1}{2}(p + 2p\cos x + p)p\sin x = p^2(1+\cos x)\sin x$

Let $\frac{dA}{dx} = 0$, .: $\cos x = \frac{1}{2}$, $x = \frac{\pi}{3}$ **D**

Q22 John: $\Pr(\text{at least once}) = 1 - \Pr(\text{none}) = 1 - \left(\frac{3}{4}\right)^4 = \frac{175}{256}$

Rebecca: $\Pr(\text{at least once}) = 1 - \Pr(\text{none}) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$

Rebecca : John $= \frac{3}{4} : \frac{175}{256} = 192 : 175$ **E**

SECTION 2

Q1a $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$

Period $= \frac{2\pi}{\frac{\pi}{3}} = 6$ months; amplitude $= 400$

Q1b Maximum population $= 1200 + 400 = 1600$; minimum population $= 1200 - 400 = 800$

Q1c $n(10) = 1200 + 400\cos\left(\frac{\pi \times 10}{3}\right) = 1000$

Q1d $n < n(10)$ for $2 < t < 4$ and $8 < t < 10$

.: 4 months out of 12, i.e. $\frac{1}{3}$

Q2a $V = \pi r^2 h = \pi\left(\frac{d}{2}\right)^2 h = 216$, $h = \frac{864}{\pi d^2}$

Q2b $S = \pi r^2 + \pi dh = \pi\left(\frac{d}{2}\right)^2 + \frac{864}{d} = \frac{\pi d^2}{4} + \frac{864}{d}$

Q2c $\frac{dS}{dd} = \frac{\pi d}{2} - \frac{864}{d^2}$

Let $\frac{dS}{dd} = 0$. $\frac{\pi d}{2} - \frac{864}{d^2} = 0$, $\pi d^3 = 1728$, $d = \frac{12}{\pi^{\frac{1}{3}}}$, $S_{\min} = 108\pi^{\frac{1}{3}}$

Q2d $h = \frac{864}{\pi d^2} = \frac{864}{\pi\left(\frac{1728}{\pi}\right)^{\frac{2}{3}}} = \frac{6}{\pi^{\frac{1}{3}}}$

Q2e $V = \pi\left(\frac{d}{2}\right)^2 h = \pi h^3$

Q2f $\frac{dV}{dt} = -10$ m^3 per year

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$, $-10 = 3\pi h^2 \frac{dh}{dt}$, .: $\frac{dh}{dt} = -\frac{10}{3\pi h^2}$

Q2g When $h = 1$, $\frac{dh}{dt} = -\frac{10}{3\pi}$, i.e. the height will be decreasing at a rate of $\frac{10}{3\pi}$ m per year.

Q2h $\frac{dV}{dt} = -10$ m^3 per year

When $h = 1$, $V = \pi h^3 = \pi$, .: Volume melted $= 216 - \pi$

Number of years taken $= \frac{216 - \pi}{10} \approx 21.3$, i.e. in year 2031

Q3a $c(t) = \frac{5}{2} t e^{-\frac{3t}{2}}$, $t \ge 0$. By CAS, $c_{\max} = 0.61$ mg per litre

Q3bi By CAS, $t \approx 0.33$ h when c first reached 0.5 mg per litre

Q3bii $t \approx 1.19$ h when c dropped back to 0.5 mg per litre

.: length of time $\approx 1.19 - 0.33 = 0.86$ h

Q3ci Average rate of change of the concentration over $\left[\frac{2}{3}, 3\right]$

$= \frac{c(3) - c\left(\frac{2}{3}\right)}{3 - \frac{2}{3}} \approx \frac{0.0833 - 0.6131}{\frac{7}{3}} \approx -0.227 \approx -0.23$

Q3cii $c'(t) = \frac{5}{2} e^{-\frac{3t}{2}} + \frac{-3}{2} \times \frac{5}{2} t e^{-\frac{3t}{2}} = \frac{5}{2} e^{-\frac{3t}{2}}\left(1 - \frac{3t}{2}\right)$ or by CAS

Let $\frac{5}{2} e^{-\frac{3t}{2}}\left(1 - \frac{3t}{2}\right) = -0.227$ . .: $t_1 \approx 0.90$, $t_2 \approx 2.12$

Q3d $n(t) = Ate^{-kt}$, $t \ge 0$; $n'(t) = Ae^{-kt}(1 - kt)$

$n(0.5) = A(0.5)e^{-k\,0.5} = 0.74$ …… (1)

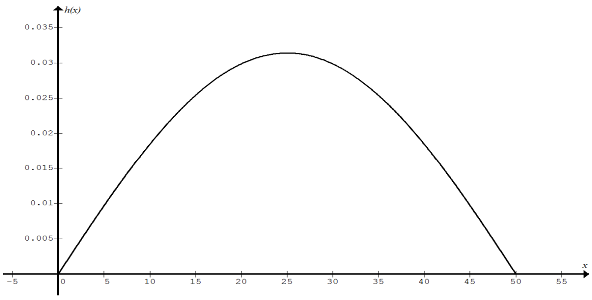
$n'(0.5) = Ae^{-k0.5}(1 - k0.5) = 0$ …. (2)

From (2), $k = 2$; from (1), $A(0.5)e^{-1} = 0.74$, .: $A = 1.48e \approx 4$

Q4a $\Pr(X \ge x) = 0.10$, $\Pr(X < x) = 0.90$, $x \approx 19.1$ cm
The minimum height is 19.1 cm approximately.

Q4b $\Pr(X < 9) \approx 0.1056$
Number of basil plants $= 2000 \times 0.1056 \approx 211$

Q4c



The mean height is 25 cm.

Q4d $\Pr(X \le x_{\max}) = 0.15$, $\int_0^{x_{\max}} \left(\frac{\pi}{100}\sin\frac{\pi x}{50}\right)dx = 0.15$,
$x_{\max} \approx 12.7$ cm

Q4e Given $p = 0.2$ is the probability that a tomato plant is tall, find the minimum number of tomato plants n so that
$\Pr(\text{at least one is tall}) > 0.95$, i.e. $\Pr(\text{no tall plants}) \le 0.05$
.: $(1-0.20)^n \le 0.05$, .: $n_{\min} = 14$

Q4fi $S \xrightarrow{0.7} S$, $S \xrightarrow{0.3} R$, $R \xrightarrow{p} R$, $R \xrightarrow{1-p} S$, $0 < p < 1$

$$\begin{matrix} & S & R \\ S & 0.7 & 1-p \\ R & 0.3 & p \end{matrix} \quad \begin{bmatrix} 0.7 & 1-p \\ 0.3 & p \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.79-0.3p \\ 0.21+0.3p \end{bmatrix}$$

.: $\Pr(\text{the third pot is smooth}) = 0.79 - 0.3p$

Q4fii $0.79 - 0.3p = 0.61$, $p = 0.6$

Q4g Given $p = 0.8$

$$\begin{matrix} S \\ R \end{matrix} \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4375 \\ 0.5625 \end{bmatrix}$$

.: $\Pr(\text{the fifth pot is smooth}) = 0.4375$

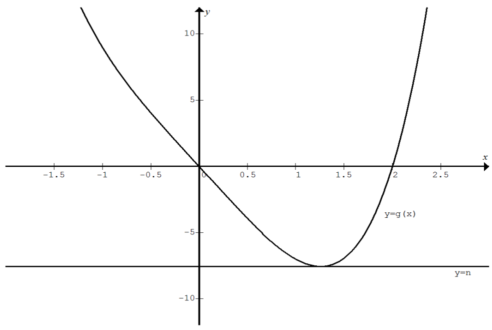
Q5a $g(x) = x^4 - 8x = x(x^3 - 2^3) = x(x-2)(x^2+2x+4)$
$= x(x-2)(x^2+2x+1-1+4) = x(x-2)((x+1)^2+3)$

Q5b Translate the graph of $y = f(x)$ to the left by 1.
$g(x) = f(x+1) = (x+1-3)(x+1-1)((x+1)^2+3)$
$= x(x-2)((x+1)^2+3)$

Q5ci $y = f(x+d) = (x+d-3)(x+d-1)((x+d)^2+3)$
For one positive x-intercept, $d-1 \ge 0$ and $d-3 < 0$
.: $d \ge 1$ and $d < 3$, i.e. $1 \le d < 3$

Q5cii For two positive x-intercepts, $d-1 < 0$ and $d-3 < 0$,
.: $d < 1$ and $d < 3$, .: $d < 1$

Q5d



The graphs of $y = g(x) = x^4 - 8x$ and $y = n$ intersect at one point when $y = n$ is a tangent to $y = g(x)$ at its minimum turning point.
$g'(x) = 4x^3 - 8$
Let $g'(x) = 0$, .: $x = 2^{\frac{1}{3}}$, $n = g\left(2^{\frac{1}{3}}\right) = -6 \times 2^{\frac{1}{3}}$

Q5ei At $(u, g(u))$, $g'(u) = m$, .: $4u^3 - 8 = m$ where $m \in R^+$
At $(v, g(v))$, $g'(v) = -m$, .: $4v^3 - 8 = -m$
.: $4u^3 + 4v^3 - 16 = 0$, .: $u^3 + v^3 = 4$

Q5eii Solve $u^3 + v^3 = 4$ and $u + v = 1$ simultaneously by CAS.
Given $g'(u) = m$ is positive, and $g'(v) = -m$ is negative, .: $u > v$
Hence $u = \frac{1+\sqrt{5}}{2}$ and $v = \frac{1-\sqrt{5}}{2}$.

Q5fi $g(p) = p^4 - 8p$, $g'(p) = 4p^3 - 8$
Equation of the tangent: $y - g(p) = g'(p)(x-p)$
$y - (p^4 - 8p) = (4p^3 - 8)(x - p)$
.: $y = (4p^3 - 8)x - 3p^4$

Q5fii Equation of the tangents: $y = (4p^3 - 8)x - 3p^4$
The tangents pass through $\left(\frac{3}{2}, -12\right)$, .: $-12 = (4p^3 - 8)\frac{3}{2} - 3p^4$
.: $3p^3(p-2) = 0$, .: $p = 0$ or $p = 2$
The tangents are: $y = -8x$ and $y = 24x - 48$

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