2014 NSW BOS Mathematics Extension 1 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
D	A	C	D	B	B	A	D	C	C

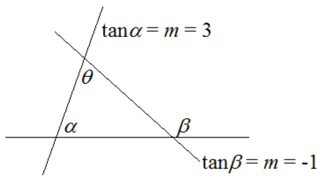
Q1 $\angle AOB = 40° \times 2 = 80°$ **D**

Q2 $\cos x - \sin x = A\cos(x+b) = A\cos x\cos b - A\sin x \sin b$

.: $A\cos b = 1$ and $A\sin b = 1$, .: $\tan b = 1$, $b = \frac{\pi}{4}$ and $A = \sqrt{2}$ **A**

Q3 The constant term is $\binom{12}{3}(2x)^9\left(-\frac{5}{x^3}\right)^3 = -\binom{12}{3}2^9 5^3$ **C**

Q4



$\theta = \beta - \alpha$, $\tan\theta = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha} = \frac{-1-3}{1-3} = 2$ **D**

Q5 Let α, β and γ be the roots.
$\alpha\beta\gamma = -42$, .: either B or D. $\alpha\beta + \beta\gamma + \gamma\alpha = -41$, .: B **B**

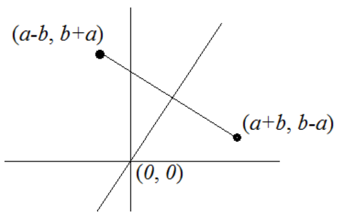
Q6 $\frac{d}{dx}\left(3\sin^{-1}\frac{x}{2}\right) = \frac{3}{\sqrt{4-x^2}}$ **B**

Q7 Period 6 and amplitude 5:
$x = 5\sin\left(\frac{\pi}{3}t\right)$, $v = \dot{x} = \frac{5\pi}{3}\cos\left(\frac{\pi}{3}t\right)$ **A**

Q8 $\binom{15}{6}5! = \frac{15!}{9!6!}5! = \frac{15!}{9!6}$ **D**

Q9 $P(x) = x^4 - 8x^3 - 7x^2 + 3 = (x^2 + x)Q(x) + ax + 3$
$P(-1) = 1 + 8 - 7 + 3 = -a + 3$, .: $a = -2$ **C**

Q10



$O(0,0)$ is equidistant from the two points $(a-b, b+a)$ and $(a+b, b-a)$, .: the locus of points (x, y) is a perpendicular bisector of the line joining the two points and passes through O.
Gradient of the line joining the two points
$= \frac{(b-a)-(b+a)}{(a+b)-(a-b)} = \frac{-a}{b}$

.: gradient of perpendicular bisector $= \frac{b}{a}$

.: the equation is $y = \frac{b}{a}x$, i.e. $bx - ay = 0$ **C**

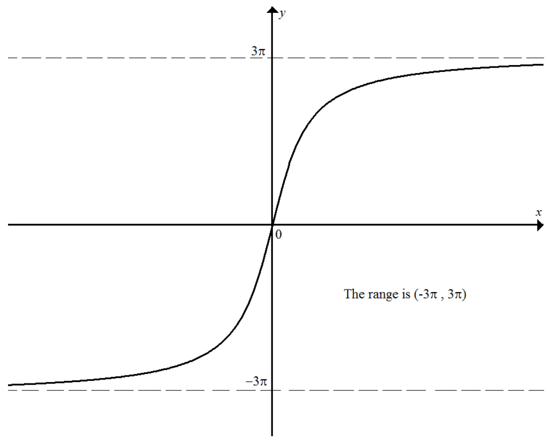
Section II

Q11a $\left(x+\frac{2}{x}\right)^2 - 6\left(x+\frac{2}{x}\right) + 9 = 0$ where $x \neq 0$, $\left(x+\frac{2}{x}-3\right)^2 = 0$,

$x + \frac{2}{x} - 3 = 0$, $x^2 - 3x + 2 = 0$, $(x-1)(x-2) = 0$.: $x = 1, 2$

Q11b $n = 30$, $p = 0.1$
$\Pr(X \le 2) = \Pr(X=0) + \Pr(X=1) + \Pr(X=2)$
$= \binom{30}{0}(0.1^0)(0.9^{30}) + \binom{30}{1}(0.1^1)(0.9^{29}) + \binom{30}{2}(0.1^2)(0.9^{28})$

Q11c



Q11d Let $x = u^2 + 1$, $x - 1 = u^2$, $\frac{dx}{du} = 2u$, $\frac{du}{dx} = \frac{1}{2u}$
When $x = 2$, $u = 1$; when $x = 5$, $u = 2$
$\int_2^5 \frac{x}{\sqrt{x-1}}dx = \int_2^5 \frac{u^2+1}{u}dx = 2\int_2^5 (u^2+1)\frac{du}{dx}dx = 2\int_1^2 (u^2+1)du$
$= 2\left[\frac{u^3}{3} + u\right]_1^2 = \frac{20}{3}$

Q11e $\frac{x^2+5}{x} > 6$, .: $x > 0$ and $x^2 - 6x + 5 > 0$,
$(x-1)(x-5) > 0$, .: $0 < x < 1$ or $x > 5$

Q11f $\frac{d}{dx}\left(\frac{e^x \ln x}{x}\right) = \frac{x\left(e^x . \frac{1}{x} + e^x \ln x\right) - e^x \ln x}{x^2}$
$= \frac{e^x(1 - \ln x + x\ln x)}{x^2}$

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Q12ai $x = 2\sin 3t$, at the start, i.e. $t = 0$, $x = 0$, the origin. When it first returns to the origin, the total distance travelled $= 2 + 2 = 4$ metres

Q12aii $v = \dot{x} = 6\cos 3t$, $a = \dot{v} = -18\sin 3t$

The particle is first at rest when $3t = \frac{\pi}{2}$, .: $a = -18$ m s^{-2}

Q12b $Volume = \int_0^{\frac{\pi}{8}} \pi y^2 dx = \int_0^{\frac{\pi}{8}} \pi\cos^2 4x\, dx = \frac{\pi}{2}\int_0^{\frac{\pi}{8}} (\cos 8x + 1)dx$

$= \frac{\pi}{2}\left[\frac{\sin 8x}{8} + x\right]_0^{\frac{\pi}{8}} = \frac{\pi}{2}\left(\frac{\pi}{8}\right) = \frac{\pi^2}{16}$ cubic units

Q12c $\ddot{x} = 2 - e^{-\frac{x}{2}}$. Let $\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 2 - e^{-\frac{x}{2}}$

.: $\frac{1}{2}v^2 = \int\left(2 - e^{-\frac{x}{2}}\right)dx = 2x + 2e^{-\frac{x}{2}} + c$

When $x = 0$, $v = 4$, .: $c = 6$ and .: $v^2 = 4\left(x + e^{-\frac{x}{2}} + 3\right)$

Q12d Use the binomial theorem to expand

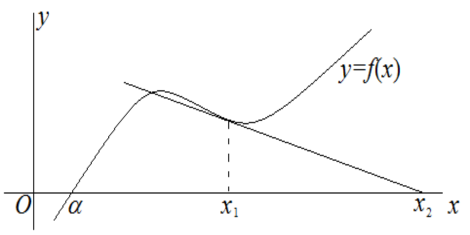
$(x+1)^n = \binom{n}{0}x^0 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$, $x \in R$

Let $x = -1$,

$(-1+1)^n = \binom{n}{0}(-1)^0 + \binom{n}{1}(-1) + \binom{n}{2}(-1)^2 + \cdots + \binom{n}{n}(-1)^n$

.: $0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}$

Q12e



x_1 is closer to α than x_2.

Q12f $T = A - Be^{-0.03t}$, $T \to 23°\text{C}$ as $t \to \infty$, .: $A = 23$ and $T = 23 - Be^{-0.03t}$.

$T = 2°\text{C}$ at $t = 0$, .: $B = 21$ and $T = 23 - 21e^{-0.03t}$

When $T = 10°\text{C}$, $10 = 23 - 21e^{-0.03t}$, $21e^{-0.03t} = 13$

.: $-0.03t = \log_e\frac{13}{21}$, $t \approx 16$, .: it takes 16 minutes approximately.

Q13a Prove $2^n + (-1)^{n+1}$ is divisible by 3, i.e. $2^n + (-1)^{n+1} = 3m$ where *m* and *n* are integers and $n \ge 1$.

When $n = 1$, $2^1 + (-1)^{1+1} = 3 = 3\times 1$, .: true

When $n = k$, assume $2^k + (-1)^{k+1} = 3m$ is true.

Consider $n = k + 1$,

$2^{k+1} + (-1)^{k+1+1} = 2\times 2^k + (-1)(-1)^{k+1} = 2\times 2^k + (2-3)(-1)^{k+1}$

$= 2\times 2^k + 2(-1)^{k+1} - 3(-1)^{k+1} = 2\left(2^k + (-1)^{k+1}\right) - 3(-1)^{k+1}$

$= 2(3m) - 3(-1)^{k+1} = 3\left(2m - (-1)^{k+1}\right)$, .: true

Hence it is true for all integers $n \ge 1$.

Q13bi $L^2 = x^2 + 40^2$, $\frac{d}{dx}L^2 = \frac{d}{dx}\left(x^2 + 40^2\right)$, $2L\frac{dL}{dx} = 2x$

$\frac{dL}{dx} = \frac{x}{L}$, .: $\frac{dL}{dx} = \cos\theta$

Q13bii $\frac{dL}{dt} = \frac{dx}{dt}\times\frac{dL}{dx} = 3\cos\theta$

Q13ci $PQ : QS = t^2 : 1$

Point Q, x coordinate $= \frac{t^2\times 0 + 1\times 2at}{t^2+1} = \frac{2at}{t^2+1}$

y coordinate $= \frac{t^2\times a + 1\times at^2}{t^2+1} = \frac{2at^2}{t^2+1}$

Q13cii Slope of $OQ = \frac{\frac{2at^2}{t^2+1}}{\frac{2at}{t^2+1}} = t$

Q13ciii Let (x, y) be Q.

From part ii, slope of $OQ = t = \frac{y}{x}$ where $x \ne 0$

Eliminate t from $x = \frac{2at}{t^2+1}$ and $t = \frac{y}{x}$:

$x = \frac{2a\frac{y}{x}}{\left(\frac{y}{x}\right)^2 + 1}$, $x = \frac{2axy}{x^2+y^2}$, .: $x^2 + y^2 = 2ay$, $x^2 + y^2 - 2ay = 0$

Completing the square: $x^2 + y^2 - 2ay + a^2 = a^2$,

$x^2 + (y-a)^2 = a^2$, .: the locus of Q is a circle centred at $(0, a)$ with radius of a units.

Q13di $\angle BAC + \angle PQB = 180°$ (Sum of opposite angles of a cyclic quadrilateral.

$\angle CQP + \angle PQB = 180°$ (Supplementary angles)

.: $\angle BAC = \angle CQP$

Q13dii $\angle CPR = \angle OPA$ (Vertically opposite angles)

$\angle OPA = \angle BAC$ (Equal angles of isosceles triangle)

$\angle BAC = \angle CQP$ from part i.

.: $\angle CPR = \angle CQP$

This result indicates that the line OP is a tangent to the circle through *P*, *Q* and *C*, yielding equal angles in alternate segments.

Q14ai $x = Vt\cos\theta$.: $t = \dfrac{x}{V\cos\theta}$

$y = -\frac{1}{2}gt^2 + Vt\sin\theta$.: $y = -\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$

.: $y = x\tan\theta - \dfrac{gx^2}{2V^2}\sec^2\theta$

Q14aii The angle between the downslope OP and the horizontal is $\frac{\pi}{4}$, .: $y = -x$ and .: $x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta = -x$

$\frac{gx^2}{2V^2}\sec^2\theta - x(1+\tan\theta) = 0$, .: $x\left(\frac{gx}{2V^2}\sec^2\theta - (1+\tan\theta)\right) = 0$

.: at point P, $\frac{gx}{2V^2}\sec^2\theta = (1+\tan\theta)$, $x = \dfrac{2V^2(1+\tan\theta)}{g\sec^2\theta}$

$\frac{x}{D} = \cos\frac{\pi}{4}$, .: $D = \sqrt{2}\,x = \dfrac{2\sqrt{2}\,V^2(1+\tan\theta)}{g\sec^2\theta}$

$D = \dfrac{2\sqrt{2}\,V^2}{g}\left(\cos^2\theta + \tan\theta\cos^2\theta\right)$

.: $D = \dfrac{2\sqrt{2}\,V^2}{g}\cos\theta(\cos\theta + \sin\theta)$

Q14aiii

$\dfrac{dD}{d\theta} = \dfrac{2\sqrt{2}\,V^2}{g}\left(-\sin\theta(\cos\theta+\sin\theta) + \cos\theta(-\sin\theta+\cos\theta)\right)$

$= \dfrac{2\sqrt{2}\,V^2}{g}\left(\cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta\right)$

$= \dfrac{2\sqrt{2}\,V^2}{g}(\cos 2\theta - \sin 2\theta)$

Q14aiv

Let $\dfrac{dD}{d\theta} = \dfrac{2\sqrt{2}\,V^2}{g}(\cos 2\theta - \sin 2\theta) = 0$ where $0 \le \theta < \dfrac{\pi}{2}$

.: $\cos 2\theta - \sin 2\theta = 0$ and .: $\theta \ne \dfrac{\pi}{4}$

.: $\tan 2\theta = 1$, $2\theta = \dfrac{\pi}{4}$, $\theta = \dfrac{\pi}{8}$

$\dfrac{d^2D}{d\theta^2} = \dfrac{2\sqrt{2}\,V^2}{g}(-2\sin 2\theta - 2\cos 2\theta)$

$= -\dfrac{4\sqrt{2}\,V^2}{g}(\sin 2\theta + \cos 2\theta)$

When $\theta = \dfrac{\pi}{8}$, $\dfrac{d^2D}{d\theta^2} < 0$, .: D has a maximum values.

Q14bi Winning on the second turn occurs when player A spins "R" on their first turn and player B loses on the second turn.

.: Pr(first win or second win)

= Pr(first win) + Pr(second win)

$= p + rq = p + r(1-r-p) = (1-r)(p+r)$

Q14bii Pr(A wins eventually)

= Pr(first win or second win or third win or $\cdots$)

$= p + rq + rrp + rrrq + rrrrp + rrrrrq + \cdots$

$= (p+rq) + (p+rq)r^2 + (p+rq)r^4 + \cdots$

$= \dfrac{p+rq}{1-r^2} = \dfrac{(1-r)(p+r)}{(1-r)(1+r)} = \dfrac{p+r}{1+r}$

Please inform mathline@itute.com re conceptual and/or mathematical errors.