2014 Mathematical Methods (CAS) Trial Exam 2 Solutions

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
A	D	E	D	E	C	A	D	B	D	D

12	13	14	15	16	17	18	19	20	21	22
A	D	C	A	B	A	A	D	C	C	A

Q1 The inverse of $y=\frac{x}{x-1}$ is $x=\frac{y}{y-1}$, $x(y-1)=y$,

$xy-x=y$, $xy-y=x$, $y=\frac{x}{x-1}$, .: $f^{-1}(x)=\frac{x}{x-1}$ A

Q2 $\log_a(1-x)=1-\log_a x$, $a>0$ and $0<x<1$

$\log_a(1-x)+\log_a x=1$, $\log_a x(1-x)=1$, $x(1-x)=a$,

$x^2-x+a=0$, $x=\frac{1\pm\sqrt{1-4a}}{2}$

.: $1-4a>0$ and $\sqrt{1-4a}<1$, .: $0<a<0.25$ D

Q3 $\frac{x+a}{x-b}\ge 0$ where $a, b\in R^+$,

either $x+a\ge 0$ and $x-b>0$, i.e. $x\ge -a$ and $x>b$, i.e. $x>b$
or $x+a\le 0$ and $x-b<0$, i.e. $x\le -a$ and $x<b$, i.e. $x\le -a$

.: the maximal domain is $(-\infty,-a]\cup(b,\infty)$ E

Q4 $f(x)=\frac{x^2-a}{x+\sqrt{a}}=\frac{(x+\sqrt{a})(x-\sqrt{a})}{x+\sqrt{a}}=x-\sqrt{a}$ for $x\ne-\sqrt{a}$

.: $f(x)\ne-\sqrt{a}-\sqrt{a}$, i.e. $-2\sqrt{a}$

The range of $f(x)$ is $R\setminus\{-2\sqrt{a}\}$ D

Q5 $y=mx-2m=m(x-2)$

$y=x^3-6x^2+8x=x(x^2-6x+8)=x(x-2)(x-4)$

The functions have the same *x*-intercepts, $x=2$

For $y=x^3-6x^2+8x$, $\frac{dy}{dx}=3x^2-12x+8=-4$ at $x=2$

.: $y=mx-2m$ intersect $y=x^3-6x^2+8x$ once only when $m\le-4$ E

Q6 $f(x)=x(x^2-1)$ C

$f(x-1)=(x-1)(x^2-2x)$

$f(1-x)=(1-x)(-2x+x^2)=-(x-1)(x^2-2x)$

.: $f(1-x)=-f(x-1)$

Q7 $\cos A+\sin B=0$

$\sin B=-\cos A=-\sin\left(\frac{\pi}{2}-A\right)=\sin\left(A-\frac{\pi}{2}\right)$, $B=A-\frac{\pi}{2}$ A

Q8 $f(x)=c-9x+6x^2-x^3$

When $c=0$, $f(x)=-9x+6x^2-x^3=-x(3-x)^2$, .: *x*-intercepts are at $x=0$ and $x=3$ (a local maximum)

Let $f'(x)=-9+12x-3x^2=-3(x-1)(x-3)=0$,

local minimum point is $(1,-4)$

For three positive x-intercepts, $c>0$ and $c<4$ D

Q9 $y=(1-x)^3+x-1\to(2-x)^3-(2-x)\to(2+x)^3-(2+x)$

$=(2+x)((2+x)^2-1)=(2+x)(1+x)(3+x)$ B

Q10 $y=g(|x+a|)$ D

For $x+a>0$, i.e. $x>-a$, $y=g(x+a)$

For $x+a<0$, i.e. $x<-a$, $y=g(-(x+a))$

The two sections are reflection of each other in the line $x=-a$.

Q11 For $x<a$, $f(x)=2x-1$, $f'(x)=2$

As $x\to a$, $f(x)\to 2a-1$, $f'(x)\to 2$

For $x\ge a$, $f(x)=ax^2+bx$, $f'(x)=2ax+b$

As $x\to a$, $f(x)\to a^3+ab$, $f'(x)\to 2a^2+b$

Since $f(x)$ is differentiable at $x=a$,

.: $a^3+ab=2a-1$ and $2a^2+b=2$.: $a=1$ D

Q12 A degree 4 polynomial may not have an inflection point, e.g. $P(x)=x^4+4x^3+6x^2+4x+1=(x+1)^4$ A

Q13 $1\le 2\sin\frac{\pi x}{2}<2$, $x\in\left[\frac{1}{3},1\right)\cup\left(1,\frac{5}{3}\right]$ D

Q14 $f(-x)=-f(x)$, .: $\int_{-1}^{1}f(-x)dx=0$

Translate by 1 unit in the positive *x*-direction,

$\int_{-1+1}^{1+1}f(-(x-1))dx=0$, i.e. $\int_{0}^{2}f(1-x)dx=0$ C

Q15 $f(x)=\sqrt{x}$, $f'(x)=\frac{1}{2\sqrt{x}}$

$\sqrt{50}\approx\sqrt{49}+1\times\frac{1}{2\sqrt{49}}=\frac{99}{14}$ A

Q16 B

Q17 $\int_{-a}^{a}\cos^2\theta\,d\theta=2\times\int_{0}^{a}\cos^2\theta\,d\theta=2\times\frac{1}{2}\int_{0}^{a}(1+\cos 2\theta)d\theta$

$=\left[\theta+\frac{\sin 2\theta}{2}\right]_0^a=a+\frac{1}{2}\sin 2a=a+\sin a\cos a$ A

Q18 $\mu=np=9$, $\sigma^2=npq=6$, .: $9q=6$, $q=\frac{2}{3}$, $p=\frac{1}{3}$

$\Pr(X>12)=\Pr(X\ge 13)\approx 0.079$ A

Q19 $\begin{matrix} & L & O \\ L & 0.15 & 0.10 \\ O & 0.85 & 0.90 \end{matrix}$

Long run probability of on time $=\dfrac{0.85}{0.10+0.85}=\dfrac{17}{19}$ D

Q20 $\int_{-4}^{0} p\,dx+\int_{1}^{9p} 4p\,dx=1$, where $p>0$, $[px]_{-4}^{0}+[4px]_{1}^{9p}=1$

$4p+36p^2-4p=1$, $p=\dfrac{1}{6}$ C

Q21 $\Pr(B'\mid A)=1-\Pr(B\mid A)=1-\dfrac{\Pr(B\cap A)}{\Pr(A)}=1-\dfrac{1}{2\Pr(A)}$ C

Q22 $2^x=5^y=100^z$, $2^x=5^y=10^{2z}$, $2^x=5^y=2^{2z}5^{2z}$

$\therefore\ x=\log_2\left(2^{2z}5^{2z}\right)=2z+2z\log_2 5$ and

$y=\log_5\left(2^{2z}5^{2z}\right)=2z\log_5 2+2z$

$\therefore\ \dfrac{x-2z}{2z}=\log_2 5$ and $\dfrac{y-2z}{2z}=\log_5 2$

$\therefore\ \dfrac{x-2z}{2z}=\dfrac{2z}{y-2z}$, $z=\dfrac{xy}{2(x+y)}$ A

SECTION 2

Q1a Translations of 3 units to the left and 4 units downwards

Q1bi $y=2$, $[-4,4]$

Q1bii $x=-4$, $[-2,2]$

Q1c $WXYZ$ is similar to $ABCD$ because the corresponding sides are dilated by the same factor, 2, e.g. $WX=2AB$ and $WZ=2AD$

Q1d $2y=x^2-6x+17$, $2y=x^2-6x+9-9+17$,

$\therefore\ 2y=(x-3)^2+8$, $2(y-4)=(x-3)^2$

$\therefore$ Translations of 3 units to the left and 4 units downwards will take the vertex of parabola ***II*** to the origin.

Q1e $2(y-4)=(x-3)^2\rightarrow 2((y+4)-4)=((x+3)-3)^2$

The new equation is $2y=x^2$.

Q1f $2y=x^2$, $\dfrac{2y}{4}=\dfrac{x^2}{4}$, $\left(\dfrac{y}{2}\right)=\left(\dfrac{x}{2}\right)^2$

$\therefore\ 2y=x^2$ is the dilations of $y=x^2$ by the same factor of 2 vertically and horizontally, $\therefore\ 2y=x^2$ and $y=x^2$ are similar.

Q1g Consider a general parabola $y=a(x-h)^2+k$. Two translations will move its vertex to the origin and its equation becomes $y=ax^2$, $\therefore\ ay=a^2x^2$, $(ay)=(ax)^2$. $\therefore\ y=ax^2$ is the dilations of $y=x^2$ by the same factor of $\dfrac{1}{a}$ vertically and horizontally. $\therefore$ all parabolas are similar to $y=ax^2$. Hence they are similar to each other.

Q2a $y=\dfrac{1}{a-b}\left(e^{-bt}-e^{-at}\right)$

When $t=\log_e 2$, $e^t=2$

$y=\dfrac{1}{a-b}\left(e^{-bt}-e^{-at}\right)=\dfrac{1}{a-b}\left((e^t)^{-b}-(e^t)^{-a}\right)=\dfrac{1}{a-b}\left(2^{-b}-2^{-a}\right)$

$=\dfrac{1}{(a-b)}\left(\dfrac{1}{2^b}-\dfrac{1}{2^a}\right)=\dfrac{1}{(a-b)}\left(\dfrac{2^a-2^b}{2^a2^b}\right)=\dfrac{2^a-2^b}{(a-b)2^{a+b}}$

Q2b $y=\dfrac{1}{a-b}\left(e^{-bt}-e^{-at}\right)$

When $t=1$, $y=\dfrac{e-1}{e^2}$, $\therefore\ \dfrac{e-1}{e^2}=\dfrac{1}{a-b}\left(e^{-b}-e^{-a}\right)$

$\therefore\ \dfrac{e-1}{e^2}=\dfrac{e^a-e^b}{(a-b)e^{a+b}}$ ……… (1)

When $t=2$, $y=\dfrac{e^2-1}{e^4}$, $\therefore\ \dfrac{e^2-1}{e^4}=\dfrac{1}{a-b}\left(e^{-2b}-e^{-2a}\right)$

$\therefore\ \dfrac{e^2-1}{e^4}=\dfrac{e^{2a}-e^{2b}}{(a-b)e^{2(a+b)}}$ ……… (2)

Q2c Factorise both sides of (2):

$\dfrac{(e-1)(e+1)}{e^2e^2}=\dfrac{(e^a-e^b)(e^a+e^b)}{(a-b)e^{a+b}e^{a+b}}$ ……… (3)

(3)/(1) $\dfrac{e+1}{e^2}=\dfrac{e^a+e^b}{e^{a+b}}$

$\therefore\ e^{-1}+e^{-2}=e^{-b}+e^{-a}$, $\therefore\ a=2$ and $b=1$

Q2di $y=\dfrac{1}{a-b}\left(e^{-bt}-e^{-at}\right)$, $\dfrac{dy}{dt}=\dfrac{1}{a-b}\left(-be^{-bt}+ae^{-at}\right)$

Q2dii For maximum y, let $\dfrac{dy}{dt}=\dfrac{1}{a-b}\left(-be^{-bt}+ae^{-at}\right)=0$

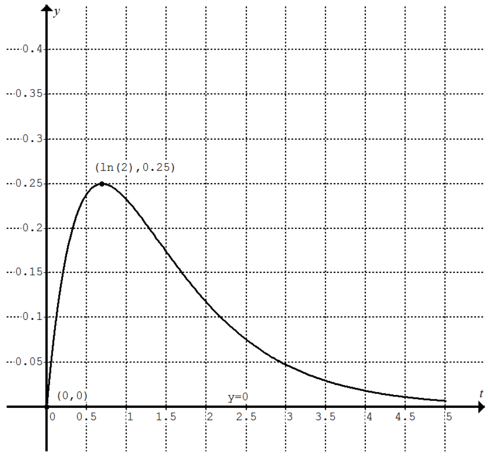
$\therefore\ ae^{-at}=be^{-bt}$, $e^{(a-b)t}=\dfrac{a}{b}$, $(a-b)t=\log_e\dfrac{a}{b}$

$\therefore\ t=\dfrac{1}{a-b}\log_e\dfrac{a}{b}$

Q2diii If $a=2$ and $b=1$, $y_{\max}$ occurs when $t=\log_e 2$

From part a, $y_{\max}=\dfrac{2^2-2^1}{(2-1)2^{2+1}}=\dfrac{1}{4}$

Q2e

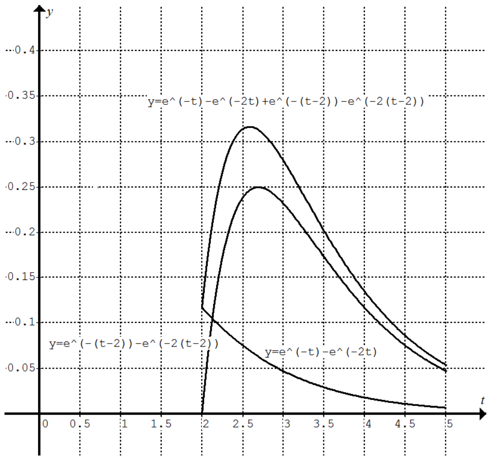


Q2f $y_{\text{first}} = e^{-t} - e^{-2t}$ for $t \geq 0$

$y_{\text{second}} = e^{-(t-2)} - 2^{-2(t-2)}$ for $t \geq 2$

$y = y_{\text{first}} + y_{\text{second}} = e^{-t} - e^{-2t} + e^{-(t-2)} - 2^{-2(t-2)}$ for $t \geq 2$

Q2g



Q2h $t \approx 3.85$ by CAS

Q3ai $\int_0^{1000} 1000(1-\cos nx)\,dx = \frac{1000^2}{3}$, $\int_0^{1000} (1-\cos nx)\,dx = \frac{1000}{3}$

$\left[x - \frac{\sin nx}{n}\right]_0^{1000} = \frac{1000}{3}$, $1000 - \frac{\sin 1000n}{n} = \frac{1000}{3}$

$\frac{\sin 1000n}{n} = \frac{2000}{3}$, $\therefore$ $3\sin 1000n = 2000n$

Q3aii Use CAS to solve $3\sin 1000n = 2000n$ for n,

$n \approx 0.001496$

$y \approx 1000(1-\cos(0.001496\,x))$

Q3bi Before reflection: $y = A\log_e\left(1 + \frac{x}{150}\right)$

After reflection: $x = A\log_e\left(1 + \frac{y}{150}\right)$, $\log_e\left(1+\frac{y}{150}\right) = \frac{x}{A}$

$1 + \frac{y}{150} = e^{\frac{x}{A}}$, $y = 150\left(e^{\frac{x}{A}} - 1\right)$

Q3bii $\int_0^{1000} 150\left(e^{\frac{x}{A}} - 1\right)dx = \frac{1000^2}{3}$, $\int_0^{1000}\left(e^{\frac{x}{A}} - 1\right)dx = \frac{1000^2}{450}$

$\left[Ae^{\frac{x}{A}} - x\right]_0^{1000} = \frac{1000^2}{450}$, $Ae^{\frac{1000}{A}} - 1000 - A = \frac{1000^2}{450}$

$A\left(e^{\frac{1000}{A}} - 1\right) = \frac{29000}{9}$, $e^{\frac{1000}{A}} = 1 + \frac{29000}{9A}$

$\therefore$ $A\log_e\left(1 + \frac{29000}{9A}\right) = 1000$

Q3biii Use CAS to solve $A\log_e\left(1+\frac{29000}{9A}\right) = 1000$ for A,

$A \approx 496.7$

$y \approx 496.7\log_e\left(1 + \frac{x}{150}\right)$

Q3c Let D metres be the northerly distance between curve ***I*** and curve ***II***.

$D \approx 496.7\log_e\left(1+\frac{x}{150}\right) - 1000(1-\cos(0.001496\,x))$

$\frac{dD}{dx} = \frac{496.7 \times \frac{1}{150}}{1 + \frac{x}{150}} - 1000(0.001496\sin(0.001496\,x))$

For longest distance, let $\frac{dD}{dx} = 0$

$\frac{496.7}{150+x} - 1.496\sin(0.001496\,x) = 0$

By CAS, $x \approx 417.633$, $D_{\text{longest}} \approx 472$ m

Q4a $\mu = 588.4$, $\sigma = 77.9$, $\Pr(X > 583.6) \approx 0.525$, i.e. 52.5%

Q4b $\mu \mp \sigma = 588.4 \mp 77.9 = 510.5,\ 666.3 \approx 511,\ 666$

Q4c $\mu = 583.6$, $\sigma = 82.2$, $\Pr(510.5 < X < 666.3) \approx 0.656$, i.e. 65.6%

Q4d Binomial distribution: $n = 10$, $p = 0.525$,

$\Pr(X > 6) \approx 0.22$

Q4e $\mu = 588.4$, $\sigma = 77.9$

$\Pr(X > 583.6 \mid X < 788) = \frac{\Pr(583.6 < X < 788)}{\Pr(X < 788)} \approx 0.522$, i.e. 52.2%

Q4f $\Pr(X < 668) = 0.80$ and $\Pr(X > 544) = 0.70$

$\Pr\left(Z < \frac{668-\mu}{\sigma}\right) = 0.80$ and $\Pr\left(Z > \frac{544-\mu}{\sigma}\right) = 0.70$

$\frac{668-\mu}{\sigma} = 0.8416$, $\frac{544-\mu}{\sigma} = -0.5244$

$\therefore$ $\mu \approx 591.6$ and $\sigma \approx 90.8$

Q4g In Vic., $\Pr(X < a) = 0.95$, $a \approx 720.47$

In NSW, $\Pr(X > 720.47) \approx 0.08$, i.e. the top 8%, $x = 8$

Q4h $\Pr(baaa) = 1 \times 0.4 \times 0.9 \times 0.9 \approx 0.32$

Q4i $\Pr(aaab) + \Pr(aaba) + \Pr(aaaa)$

$= 1\times1\times0.9\times0.1 + 1\times1\times0.1\times0.4 + 1\times1\times0.9\times0.9 \approx 0.94$

Q4j $\Pr(aaaa) = 1\times0.9\times1\times0.9 = 0.81$

$\Pr(aaab) + \Pr(abaa) = 1\times0.9\times1\times0.1 + 1\times0.1\times1\times0.9 = 0.18$

$\Pr(abab) = 1\times0.1\times1\times0.1 = 0.01$

$E(X) = 4\times0.81 + 3\times0.18 + 2\times0.01 = 3.8$

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