2013 VCAA Math Methods CAS Exam 1 Solutions

Q1a $y = x^2 \log_e x$, $\frac{dy}{dx} = (x^2)\left(\frac{1}{x}\right) + (2x)(\log_e x) = x(1 + 2\log_e x)$

Q1b $f(x) = e^{x^2}$, $f'(x) = 2xe^{x^2}$, $f'(3) = 6e^9$

Q2 $\int (4-2x)^{-5}\,dx = \frac{(4-2x)^{-4}}{(-4)(-2)} = \frac{1}{8(2x-4)^4} = \frac{1}{128(x-2)^4}$

Q3 $g'(x) = \sin(2\pi x)$, $g(x) = \int \sin(2\pi x)dx = \frac{-\cos(2\pi x)}{2\pi} + c$

$g(1) = \frac{-\cos(2\pi)}{2\pi} + c = \frac{1}{\pi}$, $-\frac{1}{2\pi} + c = \frac{1}{\pi}$, $c = \frac{3}{2\pi}$

$g(x) = \frac{3 - \cos(2\pi x)}{2\pi}$

Q4 $\sin\left(\frac{x}{2}\right) = -\frac{1}{2}$, $2\pi \le x \le 4\pi$, i.e. $\pi \le \frac{x}{2} \le 2\pi$

$\frac{x}{2} = \frac{7\pi}{6}, \frac{11\pi}{6}$ $\therefore x = \frac{7\pi}{3}, \frac{11\pi}{3}$

Q5a $2\log_3(5) - \log_3(2) + \log_3(x) = 2$,

$\log_3(25) - \log_3(2) + \log_3(x) = 2$, $\log_3\left(\frac{25x}{2}\right) = 2$

$\therefore \frac{25x}{2} = 3^2$, $x = \frac{18}{25}$

Q5b $3^{-4x} = 9^{6-x}$, $3^{-4x} = (3^2)^{6-x}$, $3^{-4x} = 3^{12-2x}$

$\therefore -4x = 12 - 2x$, $x = -6$

Q6 $g(x) = (a-x)^2 = (x-a)^2$

Average of g on $[-1, 1] = \frac{\int_{-1}^{1} (x-a)^2\,dx}{1-(-1)} = \frac{31}{12}$

$\left[\frac{(x-a)^3}{3}\right]_{-1}^{1} = \frac{31}{6}$, $\left[(x-a)^3\right]_{-1}^{1} = \frac{31}{2}$, $(1-a)^3 - (-1-a)^3 = \frac{31}{2}$

$((1-a)-(-1-a))((1-a)^2 + (1-a)(-1-a) + (-1-a)^2) = \frac{31}{2}$

$2(1 - 2a + a^2 - 1 + a^2 + 1 + 2a + a^2) = \frac{31}{2}$

$1 + 3a^2 = \frac{31}{4}$, $a^2 = \frac{9}{4}$, $a = \pm\frac{3}{2}$

Q7a $0.2 + 0.6p^2 + 0.1 + 1 - p + 0.1 = 1$, $0.6p^2 - p + 0.4 = 0$

$\therefore 3p^2 - 5p + 2 = 0$, $(3p-2)(p-1) = 0$, $\therefore p = \frac{2}{3}$ or $p = 1$

Q7bi $E(X) = 0 \times 0.2 + 1 \times 0.6p^2 + 2 \times 0.1 + 3(1-p) + 4 \times 0.1$

When $p = \frac{2}{3}$, $E(X) = 0.6\left(\frac{2}{3}\right)^2 + 0.2 + 3\left(1 - \frac{2}{3}\right) + 0.4 = \frac{28}{15}$

Q7bii $\Pr\left(X \ge \frac{28}{15}\right) = 0.1 + \frac{1}{3} + 0.1 = \frac{8}{15}$

Q8 $E(X) = \int_0^2 \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx$

Given $\frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right) = \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)$

$\therefore \int_0^2 \frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right)dx = \int_0^2 \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx + \int_0^2 \sin\left(\frac{\pi x}{4}\right)dx$

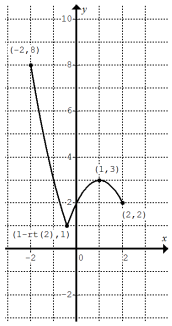
$\therefore \left[x\sin\left(\frac{\pi x}{4}\right)\right]_0^2 = E(X) + \left[-\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)\right]_0^2$

$\therefore 2 = E(X) - \frac{4}{\pi}(0-1)$, $E(X) = 2 - \frac{4}{\pi}$

Q9a Let $(x-1)^2 - 2 = 0$ where $-2 \le x \le 2$

$(x-1)^2 = 2$, $x - 1 = -\sqrt{2}$, $x = 1 - \sqrt{2}$, $\therefore a = 1 - \sqrt{2}$

Q9b



Q9ci $\left|(x-1)^2 - 2\right| + 1 \to \left|x^2 - 2\right| + 1 \to \left|x^2 - 2\right| \to \frac{1}{3}\left|x^2 - 2\right|$

Q9cii Domain of the image of g: $[-2-1, 2-1] = [-3, 1]$

Q10a $A = \frac{1}{2} \times x \times f(x) = xe^{-\frac{x}{5}}$ square units

Q10b Let $\frac{dA}{dx} = e^{-\frac{x}{2}} + x\left(-\frac{1}{5}e^{-\frac{x}{5}}\right) = e^{-\frac{x}{2}}\left(1 - \frac{x}{5}\right) = 0$

$\therefore x = 5$ and $A_{\max} = 5e^{-1}$ square units

Q10c When $y = 2e^{-\frac{x}{5}} = \frac{1}{2}$, $-\frac{x}{5} = -\log_e 4$, $x = 5\log_e 4$

When $x = 0$, $y = 2$

Area of shaded region $= \frac{1}{2}\left(2 + \frac{1}{2}\right) \times 5\log_e 4 - \int_0^{5\log_e 4} 2e^{-\frac{x}{5}}dx$

$= \frac{25}{2}\log_e 2 - \left[-10e^{-\frac{x}{5}}\right]_0^{5\log_e 4} = \frac{25}{2}\log_e 2 - \frac{15}{2} = \frac{5}{2}(5\log_e 2 - 3)$

square units

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors