2012 NSW BOS Mathematics Extension 1 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
C	A	B	C	D	A	C	D	D	B

Q1 $x^3-27=x^3-3^3=(x-3)(x^2+3x+9)$ C

Q2 $P(x,y)$, $x=-2+\frac{3}{5}(8-(-2))=4$, $y=2+\frac{3}{5}(-3-2)=-1$,
$\therefore P(4,-1)$ A

Q3 $x^3+bx^2+cx+d=(x-\alpha)(x-\beta)(x-\gamma)=0$, where
$b=-(\alpha+\beta+\gamma)=-(-2)=2$, $c=\alpha\beta+\beta\gamma+\gamma\alpha=3$ and
$d=-\alpha\beta\gamma=-1$ B

Q4 The graph is the result of $y=\sin^{-1}x$ undergoing two transformations: vertical dilation by a factor of 3 and a horizontal dilation by a factor of 2. $\therefore$ $y=3\sin^{-1}\frac{x}{2}$ C

Q5 $6!\times 2!=2\times 6!$ D

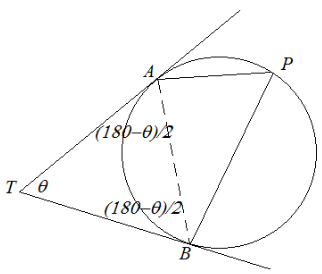
Q6 Compare $v^2=16(9-x^2)$ with $v^2=n^2(a^2-x^2)$.
$a=3$, $\therefore$ *amplitude* $A=3$; $n=4$, $\therefore$ *period* $T=\frac{2\pi}{n}=\frac{\pi}{2}$ A

Q7 Apply the double angle formula: $\cos 2A=1-2\sin^2 A$,
$\sin^2 A=\frac{1}{2}(1-\cos 2A)$
$\int \sin^2 3x dx=\int \frac{1}{2}(1-\cos 6x)dx=\frac{1}{2}\left(x-\frac{1}{6}\sin 6x\right)+C$ C

Q8 $P(x)=(x+1)(x-3)+2x+7$. When $P(x)$ is divided by $x-3$, *the remainder* $=P(3)=2\times 3+7=13$. D

Q9 $\frac{d}{dx}\cos^{-1}(3x)=\frac{-1}{\sqrt{1-(3x)^2}}\times 3=\frac{-3}{\sqrt{1-9x^2}}$ D

Q10



Angles in alternate segments are equal,
$\therefore \angle APB=\angle TAB=\frac{180°-\theta}{2}=90°-\frac{\theta}{2}$ B

Section II

Q11a $\int_0^3 \frac{1}{9+x^2}dx=\frac{1}{3}\int_0^3\frac{3}{3^2+x^2}dx=\frac{1}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^3$
$=\frac{1}{3}\tan^{-1}1=\frac{1}{3}\times\frac{\pi}{4}=\frac{\pi}{12}$

Q11b $\frac{d}{dx}(x^2\tan x)=x^2\sec^2 x+2x\tan x$

Q11c $\frac{x}{x-3}<2$
When $x>3$, $x<2(x-3)$, $x>6$.
When $x<3$, $x>2(x-3)$, $x<6$.
$\therefore$ $x<3$ or $x>6$

Q11d $\int_1^2 x(2-x)^5dx=\int_1^0-(2-u)u^5du$
$=\int_0^1(2u^5-u^6)du$
$=\left[\frac{2u^6}{6}-\frac{u^7}{7}\right]_0^1=\frac{1}{3}-\frac{1}{7}=\frac{4}{21}$

Let $u=2-x$,
$x=2-u$
$dx=-du$
When $x=1$, $u=1$
When $x=2$, $u=0$

Q11e ${}^8C_3\times{}^{10}C_4=11760$

Q11fi Let the constant term in the expansion of $\left(2x^3-\frac{1}{x}\right)^{12}$ be
${}^{12}C_m(2x^3)^m\left(-\frac{1}{x}\right)^{12-m}={}^{12}C_m(2x^3)^m(-1)^{12-m}x^{m-12}$
$\therefore$ $3m+m-12=0$, $\therefore$ $m=3$
$\therefore$ the constant term is ${}^{12}C_3(2^3)(-1)^9=-1760$.

Q11fii Let the constant term in the expansion of $\left(2x^3-\frac{1}{x}\right)^n$ be
${}^nC_m(2x^3)^m\left(-\frac{1}{x}\right)^{n-m}={}^nC_m(2x^3)^m(-1)^{n-m}x^{m-n}$
$\therefore$ $3m+m-n=0$, $\therefore$ $n=4m$ where $m=0, 1, 2, \ldots\ldots$

Q12a Let $f(n)=2^{3n}-3^n$, $n\geq 1$.
$f(1)=2^3-3=5$ is divisible by 5.
Assume $f(k)=2^{3k}-3^k$ is divisible by 5.
$f(k+1)=2^{3(k+1)}-3^{k+1}=8\times 2^{3k}-3\times 3^k=5\times 2^{3k}+3(2^{3k}-3^k)$
$\therefore$ $f(k+1)$ is divisible by 5.
Hence $f(n)=2^{3n}-3^n$ is divisible by 5 for all $n\geq 1$ by induction.

Q12bi $f(x)=\sqrt{4x-3}$, which has a range of $[0,\infty)$, is defined over R when $4x-3\geq 0$, i.e. $x\geq\frac{3}{4}$. The domain is $\left[\frac{3}{4},\infty\right)$.

Q12bii Equation of the inverse function is $x=\sqrt{4y-3}$,

$\therefore\ 4y-3=x^2$, $y=\frac{1}{4}\left(x^2+3\right)$,

$\therefore\ f^{-1}(x)=\frac{1}{4}\left(x^2+3\right)$ for $x\in[0,\infty)$.

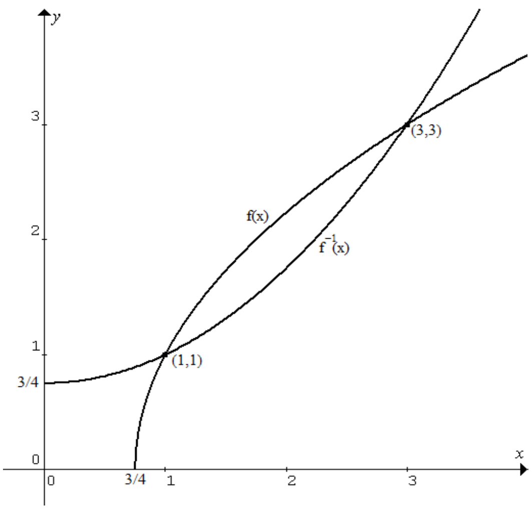
Q12biii $y=\sqrt{4x-3}$ and $y=x$

$\therefore\ x=\sqrt{4x-3}$, $x^2=4x-3$, $x^2-4x+3=0$, $(x-1)(x-3)=0$

$\therefore\ x=1$ and $y=1$ OR $x=3$ and $y=3$

The two points of intersection are $(1,1)$ and $(3,3)$.

Q12biv



Q12ci $\Pr(win)=\Pr(lose)$, $\Pr(draw)=\frac{5}{25}=\frac{1}{5}$

$\therefore\ \Pr(win)=\frac{1}{2}\left(1-\frac{1}{5}\right)=\frac{2}{5}$

Q12cii Binomial: $n=6$, $p=\frac{2}{5}$, $q=1-\frac{2}{5}=\frac{3}{5}$

$\Pr(wins.exactly.3.games)={}^6C_3\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^3=\frac{864}{3125}\approx 0.28$

Q12di $P(x,y)=P(t,y)$, $x=t$

$BC\perp AC$, $\therefore\ m_{BC}\times m_{AC}=-1$, $\frac{y}{-t}\times\frac{k}{t}=-1$, $\therefore\ y=\frac{t^2}{k}$

Q12dii The locus of P is the parabola $y=\frac{x^2}{k}$,

i.e. $x^2=ky=4\left(\frac{k}{4}\right)y$, $\therefore\ \left(0,\frac{k}{4}\right)$ is the focus.

Q13a Let $\phi=\cos^{-1}\left(\frac{2}{3}\right)$, $0<\phi<\frac{\pi}{2}$, $\therefore\ \cos\phi=\frac{2}{3}$, $\sin\phi=\frac{\sqrt{5}}{3}$

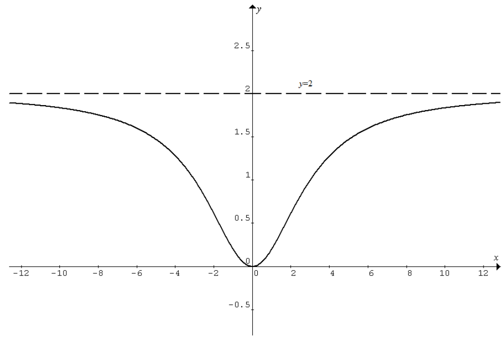
$\sin\left(2\cos^{-1}\left(\frac{2}{3}\right)\right)=\sin 2\phi=2\sin\phi\cos\phi=2\times\frac{2}{3}\times\frac{\sqrt{5}}{3}=\frac{4\sqrt{5}}{9}$

Q13bi $y=\frac{2x^2}{x^2+9}=\frac{2\left(x^2+9\right)-18}{x^2+9}=2-\frac{18}{x^2+9}$

$\therefore\ y=2$ is the horizontal asymptote.

Q13bii When $x=0$, $y=0$.

As $x\to-\infty$, $y\to 2^-$, as $x\to\infty$, $y\to 2^-$.



Q13ci $x=5+6\cos 2t+8\sin 2t$

$\dot{x}=-12\sin 2t+16\cos 2t$, $\ddot{x}=-24\cos 2t-32\sin 2t$.

$\therefore\ \ddot{x}=-4(5+6\cos 2t+8\sin 2t-5)=-2^2(x-5)$,

$\therefore$ simple harmonic motion.

Q13cii Let $A\cos(2t-\varepsilon)=6\cos 2t+8\sin 2t$.

$\therefore\ A\cos 2t\cos\varepsilon+A\sin 2t\sin\varepsilon=6\cos 2t+8\sin 2t$

$\therefore\ A\cos\varepsilon=6$ and $A\sin\varepsilon=8$

$\therefore\ A^2\cos^2\varepsilon+A^2\sin^2\varepsilon=100$, $\therefore\ A=10$

$\frac{\sin\varepsilon}{\cos\varepsilon}=\tan\varepsilon=\frac{8}{6}$, $\therefore\ \varepsilon=\tan^{-1}\frac{4}{3}$

$\therefore\ x=5+6\cos 2t+8\sin 2t=5+10\cos\left(2t-\tan^{-1}\frac{4}{3}\right)$

When $x=0$, $5+10\cos\left(2t-\tan^{-1}\frac{4}{3}\right)=0$,

$\cos\left(2t-\tan^{-1}\frac{4}{3}\right)=-\frac{1}{2}$, $\therefore\ 2t-\tan^{-1}\frac{4}{3}=\frac{2\pi}{3}$,

$t=\frac{1}{2}\left(\frac{2\pi}{3}+\tan^{-1}\frac{4}{3}\right)\approx 1.51\,\mathrm{s}$

Q13di $C(t)=1.4te^{-0.2t}$

$\frac{dC}{dt}=1.4e^{-0.2t}+1.4te^{-0.2t}(-0.2)=1.4e^{-0.2t}(1-0.2t)$

The maximum is reached when $\frac{dC}{dt}=0$,

i.e. $1.4e^{-0.2t}(1-0.2t)=0$, $\therefore\ 1-0.2t=0$, $t=5$ hours after the drug was administered.

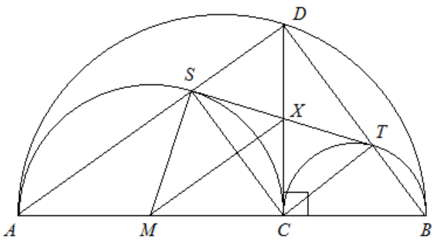
Q13dii Newton's method: $t \approx t_1 + \dfrac{C(t) - C(t_1)}{C'(t_1)}$ where

$C(t) = 1.4te^{-0.2t}$ and $C'(t) = 1.4e^{-0.2t}(1 - 0.2t)$

When $t_1 = 20$, $C(20) \approx 0.5128$, $C'(20) = -0.0769$.

When $C(t) = 0.3$, $t \approx 20 + \dfrac{0.3 - 0.5128}{-0.0769} \approx 22.8$ hours

Q14ai



$\angle ADB = 90°$, angle on the circumference subtended by the diameter

Similarly, $\angle DTC = \angle CTB = 90°$ and $\angle DSC = \angle CSA = 90°$

.: *CTDS* is a rectangle.

Q14aii Since *CTDS* is a rectangle, .: *X* is the midpoint of the diagonals, .: $SX = CX$. $MS = MC =$ *radius of the semicircle*. MX is a common side. .: ΔMXS and ΔMXC are congruent.

Q14aiii Since ΔMXS and ΔMXC are congruent, .: $\angle MSX = \angle MCX = 90°$. SM is a radius of the circle, .: ST is a tangent to the circle.

Q14bi and ii From the given diagram, $0° < \theta < 90°$

$x = 70t\cos\theta$, .: $t = \dfrac{x}{70\cos\theta}$

$y = 70t\sin\theta - 4.9t^2 = x\tan\theta - \dfrac{x^2}{1000\cos^2\theta}$, a parabola.

$y_{\max}$ occurs when $x = -\dfrac{b}{2a} = \dfrac{-\tan\theta}{2 \times \dfrac{-1}{1000\cos^2\theta}}$

$= 500\sin\theta\cos\theta = 250\sin 2\theta$

.: $y_{\max} = 500\sin\theta\cos\theta \times \tan\theta - \dfrac{(500\sin\theta\cos\theta)^2}{1000\cos^2\theta}$

$= 500\sin^2\theta - 250\sin^2\theta = 250\sin^2\theta$

Alternative method: By calculus, $\dfrac{dy}{dt} = 0$, $70\sin\theta - 9.8t = 0$,

$t = \dfrac{70\sin\theta}{9.8}$, $y_{\max} = 250\sin^2\theta$ and $x = 250\sin 2\theta$.

Q14biii For best viewing, $y \ge 150$ and $125 < x < 180$

.: $250\sin^2\theta \ge 150$, i.e. $\theta \ge 50.8°$

and $125 < 250\sin 2\theta < 180$, i.e. $15° < \theta < 23°$ or $67° < \theta < 75°$

.: $67° < \theta < 75°$

Q14ci $BG = u\cos\alpha$, $PG = u\sin\alpha$

$AG = \sqrt{1^2 + BG^2 - 2(1)(BG)\cos 60°} = \sqrt{1 + u^2\cos^2\alpha - u\cos\alpha}$

$r = \sqrt{AG^2 + PG^2} = \sqrt{1 + u^2\cos^2\alpha - u\cos\alpha + u^2\sin^2\alpha}$

$= \sqrt{1 + u^2(\cos^2\alpha + \sin^2\alpha) - u\cos\alpha}$

$= \sqrt{1 + u^2 - u\cos\alpha}$

Q14cii Given *constant speed* $\dfrac{du}{dt} = 360$ km/h

At $t = 5$ min $= \dfrac{5}{60}$ h, *distance* $u = 360 \times \dfrac{5}{60} = 30$ km

Rate $= \dfrac{dr}{dt} = \dfrac{dr}{du} \times \dfrac{du}{dt} = \dfrac{2u - \cos\alpha}{2\sqrt{1 + u^2 - u\cos\alpha}} \times \dfrac{du}{dt}$

$= \dfrac{60 - \cos\alpha}{2\sqrt{1 + 900 - 30\cos\alpha}} \times 360$

$= \dfrac{180(60 - \cos\alpha)}{\sqrt{901 - 30\cos\alpha}}$ km/h

Please inform mathline@itute.com re conceptual and/or mathematical errors.