2011 Specialist Maths Trial Exam 2 Solutions

Section 1

1	2	3	4	5	6	7	8	9	10	11
D	B	C	C	B	A	E	E	A	A	D

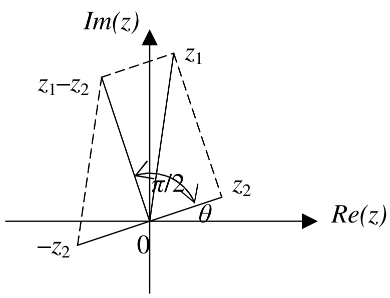
12	13	14	15	16	17	18	19	20	21	22
C	A	B	D	A	C	B	E	A	D	A

Q1 $z_1^{\,2} + 3z_2^{\,2} = 0$, $z_1 = \pm i\sqrt{3} z_2$

$\left|z_1 - z_2\right| = \left|\pm i\sqrt{3} z_2 - z_2\right| = \left|\left(\pm i\sqrt{3} - 1\right) z_2\right| = \left|\pm i\sqrt{3} - 1\right|\left|z_2\right| = 2\left|z_2\right|$

D

Q2



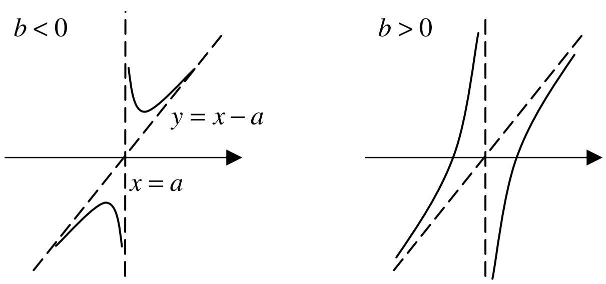
B

Q3 $P(z) = z^3 + 3iz^2 - 3z - i = (z+i)^3$ C

Q4 Let $z = x + iy$, $\text{Im}(z) = |z - i|$, $(\text{Im}(z))^2 = |z - i|^2$

$y^2 = x^2 + (y-1)^2$, $y = \frac{1}{2}x^2 + \frac{1}{2}$ C

Q5



B

Q6 For $y = \dfrac{2}{a + bx + 4ax^2}$ to have only one asymptote, the discriminant of $a + bx + 4ax^2$ must be a negative value.

.: $\Delta = b^2 - 16a^2 < 0$, .: $-4a < b < 4a$

Since $a > 1$, $-1 \le b \le 4$ satisfies the requirement $-4a < b < 4a$.

A

Q7 The equation of the hyperbola is $\dfrac{(x+1)^2}{1^2} - \dfrac{(y-2)^2}{b^2} = 1$.

The gradients of the asymptotes are $\pm b = \pm 2$ (best approximation determined from the scaled graph).

Equations of the asymptotes: $y - 2 = \pm 2(x+1)$,

i.e. $y = -2x$, $y = 2x + 4$ E

Q8 $\tan^{-1}(a) = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$

$\sin^{-1}(b) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

.: $\sec\left(\tan^{-1}(a) + \sin^{-1}(b)\right) = \sec\left(-\frac{2\pi}{3}\right) = \dfrac{1}{\cos\left(-\frac{2\pi}{3}\right)} = -2$ E

Q9 The values of a and b do not change the range of f. The range of $\cos^{-1}(x)$ is $[0, \pi]$, .: the range of $\cos^{-1}\left(\frac{x}{a} + b\right) + c$ is $[c, \pi + c]$. A

Q10 $\tan^{-1}(x - a + 1) = \tan^{-1}(x - a) + \frac{\pi}{4}$

$\tan^{-1}(x - a + 1) - \tan^{-1}(x - a) = \frac{\pi}{4}$

$\tan\left(\tan^{-1}(x - a + 1) - \tan^{-1}(x - a)\right) = \tan\left(\frac{\pi}{4}\right)$

$\dfrac{(x - a + 1) - (x - a)}{1 + (x - a + 1)(x - a)} = 1$, .: $1 = 1 + (x - a + 1)(x - a)$

.: $(x - a + 1)(x - a) = 0$, .: $x = a - 1$ or $x = a$ A

Q11 $\overrightarrow{OP} = \tilde{i} - \tilde{j}$, $\overrightarrow{OQ} = -\tilde{j} + \tilde{k}$, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -\tilde{i} + \tilde{k}$

.: $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{OQ}\right| = \left|\overrightarrow{OP}\right|$

.: ΔOPQ is equilateral.

.: $\angle OPQ = 60°$ D

Q12 $2\tilde{i} + p\tilde{j} + 3\tilde{k}$, $-\tilde{i} + 3\tilde{j} + q\tilde{k}$ and $\tilde{i} - \tilde{j} + \tilde{k}$ are linearly dependent.

$2\tilde{i} + p\tilde{j} + 3\tilde{k} + m\left(-\tilde{i} + 3\tilde{j} + q\tilde{k}\right) + n\left(\tilde{i} - \tilde{j} + \tilde{k}\right) = 0$, $m, n \ne 0$

.: $2 - m + n = 0$ .... (1)

$p + 3m - n = 0$ .... (2)

$3 + qm + n = 0$ .... (3)

(1) + (2): $m = \dfrac{-p-2}{2}$

(2) + (3): $m = \dfrac{-p-3}{q+3}$

.: $\dfrac{-p-2}{2} = \dfrac{-p-3}{q+3}$

.: $q = \dfrac{-p}{p+2}$ C

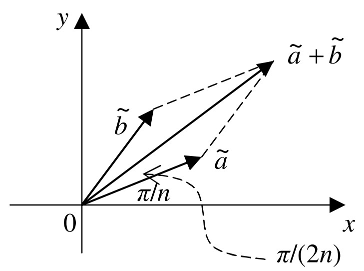
Q13 $\angle ACB = \angle ADB$

$\cos\angle ACB = \cos\angle ADB$

$\dfrac{\overrightarrow{AC}.\overrightarrow{BC}}{\left|\overrightarrow{AC}\right|\left|\overrightarrow{BC}\right|} = \dfrac{\overrightarrow{AD}.\overrightarrow{BD}}{\left|\overrightarrow{AD}\right|\left|\overrightarrow{BD}\right|}$ A

Q14 $\tilde{a} = \cos\left(\frac{\pi}{n}\right)\tilde{i} + \sin\left(\frac{\pi}{n}\right)\tilde{j}$, $\tilde{b} = \cos\left(\frac{2\pi}{n}\right)\tilde{i} + \sin\left(\frac{2\pi}{n}\right)\tilde{j}$

$\tilde{a}$ and $\tilde{b}$ have the same magnitude.



Angle between vectors $\tilde{a}+\tilde{b}$ and $\tilde{i}$ is $\frac{\pi}{n}+\frac{\pi}{2n}=\frac{3\pi}{2n}$. B

Q15 $v(t) = 2\sin^{-1}\left(\frac{t}{10}-1\right)+\pi$, $0 \le t \le 10$

$\textit{Average velocity} = \frac{\textit{displacement}}{\textit{time}}$

$$= \frac{\int_0^{10}\left(2\sin^{-1}\left(\frac{t}{10}-1\right)+\pi\right)dt}{10} = 2$$ D

Q16 $\int_0^1 \frac{1-2x-x^2}{\sqrt{1-x^2}}dx = \int_0^1 \frac{1-x^2}{\sqrt{1-x^2}}dx + \int_0^1 \frac{-2x}{\sqrt{1-x^2}}dx$

$= \int_0^1 \sqrt{1-x^2}\,dx + \int_0^1 \frac{-2x}{\sqrt{1-x^2}}dx$

$= \frac{\pi}{4} - 2$ A

Q17 $\tilde{r} = 2\cos^{-1}(t)\tilde{i} - 2\cos^{-1}(t)\tilde{j} + \cos^{-1}(t)\tilde{k}$, $0 \le t \le 1$

$\tilde{r} = \left(2\tilde{i} - 2\tilde{j} + \tilde{k}\right)\cos^{-1}(t)$

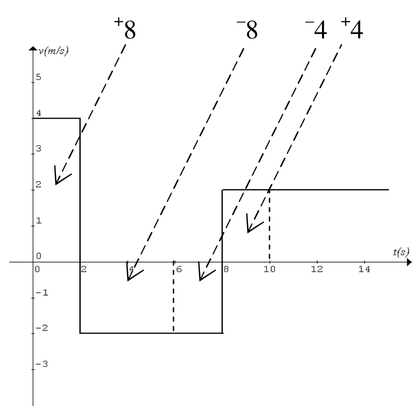
$\tilde{v} = \left(2\tilde{i} - 2\tilde{j} + \tilde{k}\right)\frac{-1}{\sqrt{1-t^2}}$

$\tilde{a} = \left(2\tilde{i} - 2\tilde{j} + \tilde{k}\right)\frac{-t}{\left(1-t^2\right)^{\frac{3}{2}}}$ C

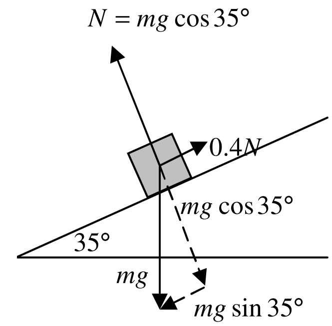
Q18 $a = {}^+2$, $u = {}^-10$, $s = {}^+16 - {}^+5 = {}^+11$, t?

$s = ut + \frac{1}{2}at^2$, $t = 11$ B

Q19 E



Q20

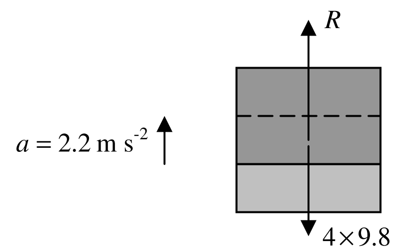


The diagram shows the particle sliding down the inclined plane. If the particle slides up the plane the force due to friction points in the opposite direction. Since $mg\sin 35° > 0.4mg\cos 35°$
.: there is always a resultant force down the plane.
.: the particle does not move at constant velocity. A

Q21 $v^2 = x-2$, $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}$

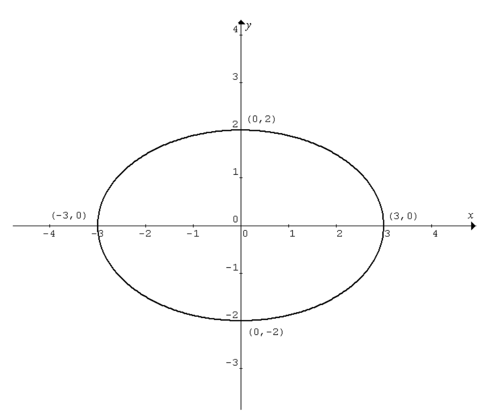
Since $v^2 = x-2 \ge 0$, .: $x \ge 2$, i.e. the particle moves along the positive x-axis.
Since $v^2 = x-2$, .: $v = \pm\sqrt{x-2}$. When $v = -\sqrt{x-2}$, the particle moves towards the origin. D

Q22 ${}^+R + 4\times{}^-9.8 = 4\times{}^+2.2$, .: $R = {}^+48$ N A



Section 2

Q1a $4x^2+9y^2=36$, .: $\frac{x^2}{9}+\frac{y^2}{4}=1$



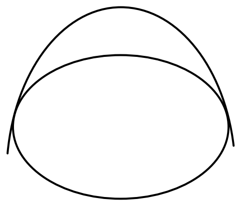
Q1b $4x^2+9y^2=36$

Implicit differentiation: $8x + 18y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{4x}{9y}$

At $y=1$, $4x^2 = 27$, .: $x = \pm\frac{3\sqrt{3}}{2}$

.: $\frac{dy}{dx} = \pm\frac{2\sqrt{3}}{3}$

Q1c $x^2+3y=c$ is an inverted parabola.
It touches the ellipse at $(0,-2)$ if $c=-6$.
It touches the ellipse at $(0,2)$ if $c=6$.
There are two other possible points:



Solve simultaneously, $x^2+3y=c$, $4x^2+9y^2=36$

.: $4(c-3y)+9y^2=36$,

.: $9y^2-12y+(4c-36)=0$

Same y-coordinate at the contact points.
To have only one y value, let the discriminant be 0.

.: $(-12)^2-4\times 9\times(4c-36)=0$

.: $c=10$

.: $9y^2-12y+4=0$, $y=\dfrac{2}{3}$, $x=\pm 2\sqrt{2}$

Two other possible points are $\left(2\sqrt{2},\dfrac{2}{3}\right)$ and $\left(-2\sqrt{2},\dfrac{2}{3}\right)$ if $c=10$.

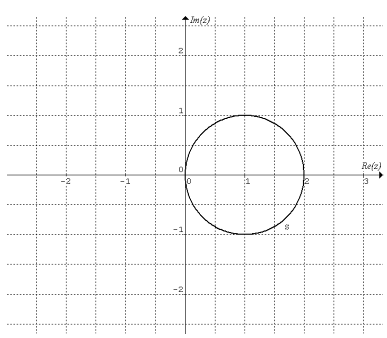
Q1di At $P(0,-2)$, $x^2+3y=-6$ is the minimum value.

Q1dii At $P\left(-2\sqrt{2},\dfrac{2}{3}\right)$ or $P\left(2\sqrt{2},\dfrac{2}{3}\right)$, $x^2+3y=10$ is the maximum value.

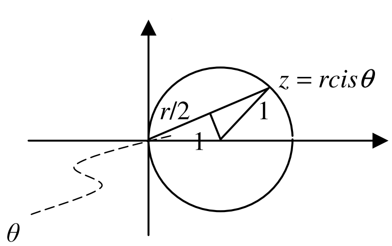
Q2ai Let $z=x+iy$

$|z-1|=1$, $|(x-1)+yi|=1$, $|(x-1)+yi|^2=1$, $(x-1)^2+y^2=1$

Q2aii



Q2bi

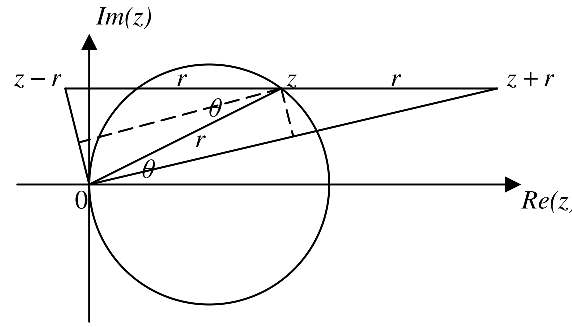


$\dfrac{r}{2}=1\cos\theta$, .: $r=2\cos\theta$

Q2bii $\dfrac{1}{z}=\dfrac{1}{rcis\theta}=\dfrac{cis(-\theta)}{r}=\dfrac{\cos(-\theta)+i\sin(-\theta)}{2\cos\theta}$

$=\dfrac{\cos\theta-i\sin\theta}{2\cos\theta}=\dfrac{1}{2}-i\dfrac{\tan\theta}{2}$

Q2biii



$Arg\left(\dfrac{z-r}{z+r}\right)=Arg(z-r)-Arg(z+r)$

$=$ *angle formed by* $z-r$, 0 and $z+r$

$=\dfrac{\pi}{2}$ because 0 is on the circumference of the circle of radius r centred at z.

Refer to the diagram: $|z-r|=2\times r\sin\dfrac{\theta}{2}$, $|z+r|=2\times r\cos\dfrac{\theta}{2}$

.: $\left|\dfrac{z-r}{z+r}\right|=\dfrac{|z-r|}{|z+r|}=\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\tan\dfrac{\theta}{2}$

Q2ci $\dfrac{z-r}{z+r}=\left|\dfrac{z-r}{z+r}\right|cis\left(Arg\left(\dfrac{z-r}{z+r}\right)\right)=\left(\tan\dfrac{\theta}{2}\right)cis\dfrac{\pi}{2}=i\tan\dfrac{\theta}{2}$

.: $\dfrac{z_1-r_1}{z_1+r_1}==i\tan\dfrac{\theta_1}{2}$, $\dfrac{z_2-r_2}{z_2+r_2}=i\tan\dfrac{\theta_2}{2}$, $\dfrac{z_3-r_3}{z_3+r_3}=i\tan\dfrac{\theta_3}{2}$

.: all three complex numbers are purely imaginary (has no real part), so they are on the imaginary axis and .: collinear.

Q2cii $\dfrac{1}{z}=\dfrac{1}{2}-i\dfrac{\tan\theta}{2}$

.: $\dfrac{1}{z_1}=\dfrac{1}{2}-i\dfrac{\tan\theta_1}{2}$, $\dfrac{1}{z_2}=\dfrac{1}{2}-i\dfrac{\tan\theta_2}{2}$, $\dfrac{1}{z_3}=\dfrac{1}{2}-i\dfrac{\tan\theta_3}{2}$

All three complex numbers have the same real part of 2, .: they line up vertically on the line $\mathrm{Re}(z)=\dfrac{1}{2}$.

Q3a $\tilde{r}(t)=\sqrt{\dfrac{1+(t-1)^2}{2}}\tilde{i}-(t-1)\tilde{j}+\dfrac{\sqrt{2}|t-2|}{4}\tilde{k}$, $t\geq 0$

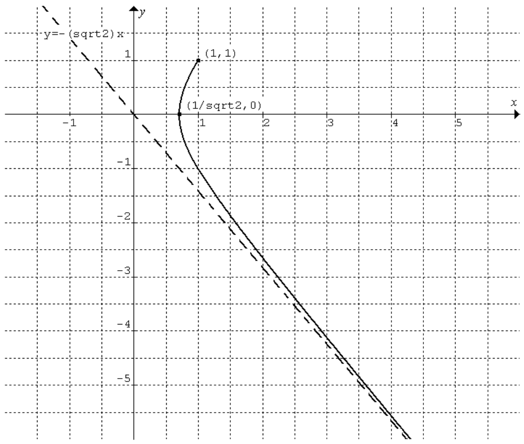
$x=\sqrt{\dfrac{1+(t-1)^2}{2}}$, $y=-(t-1)$

$x^2=\dfrac{1+(t-1)^2}{2}$, $y^2=(t-1)^2$

.: $x^2=\dfrac{1+y^2}{2}$, $2x^2-y^2=1$

.: $\dfrac{x^2}{\left(\frac{1}{\sqrt{2}}\right)^2}-y^2=1$

Q3b When $t=0$, the shadow is at $(1,1)$.



Q3c $|\tilde{r}|^2=\dfrac{1+(t-1)^2}{2}+(t-1)^2+\dfrac{(t-2)^2}{8}$

$|\tilde{r}|^2=\dfrac{1}{2}+\dfrac{3(t-1)^2}{2}+\dfrac{(t-2)^2}{8}$

Let $\dfrac{d}{dt}|\tilde{r}|^2=0$. .: $3(t-1)+\dfrac{t-2}{4}=0$

.: $t=\dfrac{14}{13}$, the time when the aeroplane was closest to the controller.

Q3di $\tilde{r}(t)=\sqrt{\dfrac{1+(t-1)^2}{2}}\,\tilde{i}-(t-1)\tilde{j}+\dfrac{\sqrt{2}|t-2|}{4}\tilde{k}$

.: $\tilde{v}=\dfrac{d}{dt}\tilde{r}=\dfrac{t-1}{2\sqrt{\dfrac{1+(t-1)^2}{2}}}\tilde{i}-\tilde{j}-\dfrac{\sqrt{2}}{4}\tilde{k}$ for $t<2$

When $t=0$, $\tilde{v}=-\dfrac{1}{2}\tilde{i}-\tilde{j}-\dfrac{\sqrt{2}}{4}\tilde{k}$

and speed $|\tilde{v}|=\sqrt{\left(-\dfrac{1}{2}\right)^2+(-1)^2+\left(-\dfrac{\sqrt{2}}{4}\right)^2}=\sqrt{\dfrac{11}{8}}=\dfrac{\sqrt{22}}{4}$

Q3dii When $t=0$, $\tilde{v}=-\dfrac{1}{2}\tilde{i}-\tilde{j}-\dfrac{\sqrt{2}}{4}\tilde{k}$

$\hat{v}=\dfrac{\tilde{v}}{|\tilde{v}|}=\sqrt{\dfrac{8}{11}}\left(-\dfrac{1}{2}\tilde{i}-\tilde{j}-\dfrac{\sqrt{2}}{4}\tilde{k}\right)$

Let θ be the angle between $\hat{v}$ and $\tilde{k}$.

$\hat{v}.\tilde{k}=-\sqrt{\dfrac{8}{11}}\times\dfrac{\sqrt{2}}{4}=-\dfrac{1}{\sqrt{11}}=\cos\theta$, .: $\theta\approx 108°$

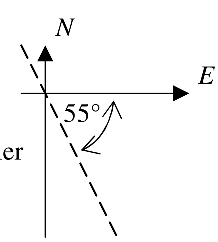
.: angle between flight path and ground $=108-90=18°$

Q3e Asymptote: $y=-\sqrt{2}x$

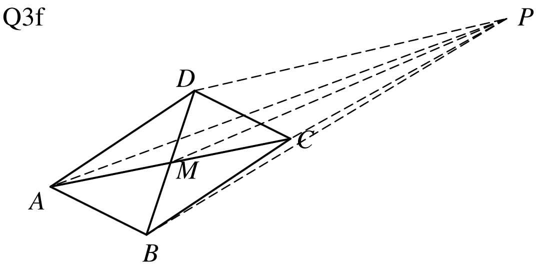
.: $\tan\phi=-\sqrt{2}$

.: $\phi\approx-55°$

True bearing of destination from controller $=90+55=145°T$



Q3f



P is at any position above the rectangle $ABCD$.

$\overrightarrow{AC}=\overrightarrow{PC}-\overrightarrow{PA}$, $\overrightarrow{BD}=\overrightarrow{PD}-\overrightarrow{PB}$

$|\overrightarrow{AC}|=|\overrightarrow{BD}|$, diagonals of rectangle $ABCD$

.: $|\overrightarrow{PC}-\overrightarrow{PA}|^2=|\overrightarrow{PD}-\overrightarrow{PB}|^2$

.: $|\overrightarrow{PC}|^2+|\overrightarrow{PA}|^2-2\times\overrightarrow{PC}.\overrightarrow{PA}=|\overrightarrow{PD}|^2+|\overrightarrow{PB}|^2-2\times\overrightarrow{PD}.\overrightarrow{PB}$ ...... (1)

M is the midpoint of the diagonals.

.: $\overrightarrow{PM}=\dfrac{1}{2}\left(\overrightarrow{PC}+\overrightarrow{PA}\right)=\dfrac{1}{2}\left(\overrightarrow{PD}+\overrightarrow{PB}\right)$

.: $|\overrightarrow{PM}|^2=\dfrac{1}{4}|\overrightarrow{PC}+\overrightarrow{PA}|^2=\dfrac{1}{4}|\overrightarrow{PD}+\overrightarrow{PB}|^2$

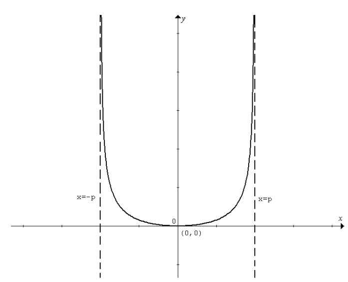
.: $|\overrightarrow{PC}|^2+|\overrightarrow{PA}|^2+2\times\overrightarrow{PC}.\overrightarrow{PA}=|\overrightarrow{PD}|^2+|\overrightarrow{PB}|^2+2\times\overrightarrow{PD}.\overrightarrow{PB}$ ...... (2)

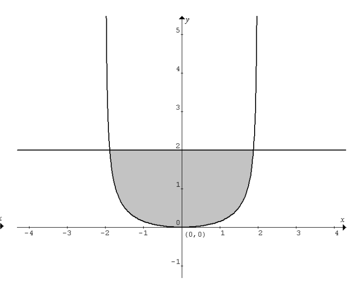
$\dfrac{(1)+(2)}{2}$: $|\overrightarrow{PA}|^2+|\overrightarrow{PC}|^2=|\overrightarrow{PB}|^2+|\overrightarrow{PD}|^2$

Q4a $f(x)=\dfrac{p}{\sqrt{p^2-x^2}}-1$, $p\in R^+$

Asymptotes: $p^2-x^2=0$, $x=\pm p$

y-intercept: $x=0$, $y=0$





Q4b $p=2$, .: $f(x)=\dfrac{2}{\sqrt{4-x^2}}-1$

At $y=2$, $\dfrac{2}{\sqrt{4-x^2}}-1=2$, $x=\pm\dfrac{4\sqrt{2}}{3}$

Area $=2\left[\dfrac{4\sqrt{2}}{3}\times 2-\int_0^{\frac{4\sqrt{2}}{3}}\left(\dfrac{2}{\sqrt{4-x^2}}-1\right)dx\right]$

$=2\left[\dfrac{8\sqrt{2}}{3}-\left[2\sin^{-1}\left(\dfrac{x}{2}\right)-x\right]_0^{\frac{4\sqrt{2}}{3}}\right]$

$=2\left[\dfrac{8\sqrt{2}}{3}-2\sin^{-1}\left(\dfrac{2\sqrt{2}}{3}\right)+\dfrac{4\sqrt{2}}{3}\right]=8\sqrt{2}-4\sin^{-1}\left(\dfrac{2\sqrt{2}}{3}\right)$

Q4ci $y=\frac{2}{\sqrt{4-x^2}}-1$, .: $x^2=4\left(1-\frac{1}{(y+1)^2}\right)$

$V=\int_0^2 \pi x^2 dy = \int_0^2 4\pi\left(1-\frac{1}{(y+1)^2}\right)dy$

Q4cii $V=4\pi\left[y+\frac{1}{y+1}\right]_0^2 = 4\pi\left(2+\frac{1}{3}-1\right)=\frac{16\pi}{3}$

Q4d Required time $=\frac{\frac{16\pi}{3}}{\frac{\pi}{3}}=16$ seconds

Q4e $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$

When the depth of water is h, volume of water V

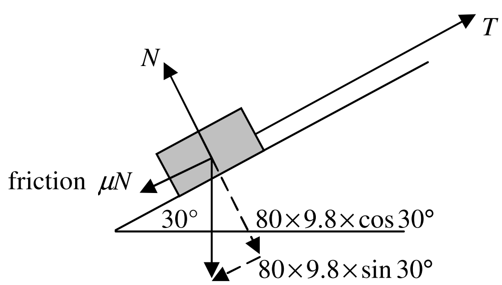
$=\int_0^h 4\pi\left(1-\frac{1}{(y+1)^2}\right)dy = 4\pi\left(h+\frac{1}{h+1}-1\right)$

.: $\frac{dV}{dh}=4\pi\left(1-\frac{1}{(h+1)^2}\right)$

When $h=1$, $\frac{\pi}{3}=3\pi\frac{dh}{dt}$

.: $\frac{dh}{dt}=\frac{1}{9}$ cm per second

Q5a

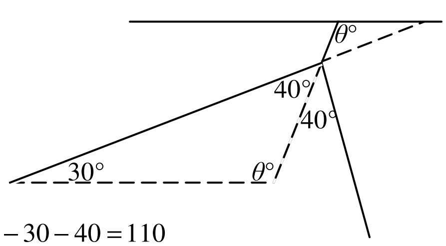


Normal reaction force of the inclined plane on the crate N
$=80\times 9.8\times\cos 30°$

Force of friction $=\mu N = 0.25\times 80\times 9.8\times\cos 30° \approx 170$ N

Q5b Applied force $=T=170+80\times 9.8\times\sin 30° \approx 562$ N

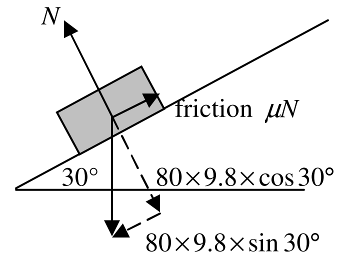
Q5ci



$\theta = 180-30-40=110$

Q5cii $T_{chain}=2\times 562\cos 40° \approx 861$ N

Q5d



Resultant force $=80\times 9.8\times\sin 30° - 170 \approx 222$ N

$a=\frac{F}{m}=\frac{222}{80}\approx 2.8$ m s^{-2}

Q5ei $u={}^{-}0.2$, $a={}^{+}2.8$, $t=0.25$, v?

$v=u+at$, $v={}^{-}0.2+{}^{+}2.8\times 0.25={}^{+}0.5$

.: speed $=0.5$ m s^{-1}

Q5eii |momentum| $=mv=80\times 0.5=40$ kg m s^{-1}

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