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2010 NSW BOS Mathematics Exam Solutions

Q1a $x^2 = 4x$, $x^2 - 4x = 0$, $x(x-4) = 0$, $x = 0$ or 4

Q1b $a + b\sqrt{5} = \dfrac{1}{\sqrt{5}-2} \times \dfrac{\sqrt{5}+2}{\sqrt{5}+2} = 2 + \sqrt{5}$, .: $a = 2$, $b = 1$

Q1c $(x+1)^2 + (y-2)^2 = 25$

Q1d $|2x+3| = 9$, $2x + 3 = -9$ or $2x + 3 = 9$, .: $x = -6$ or 3

Q1e By the product rule:

$$\frac{d}{dx}\left(x^2 \tan x\right) = \left(\frac{d}{dx}x^2\right)(\tan x) + x^2 \frac{d}{dx}\tan x = 2x \tan x + x^2 \sec^2 x$$

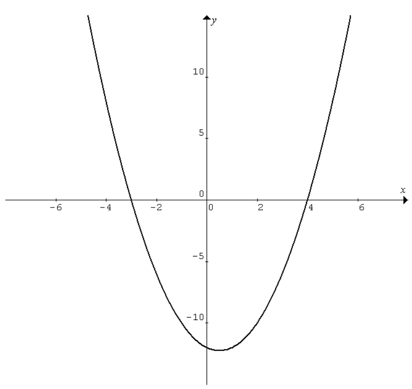
Q1f Geometric series: $a = 1$, $r = -\dfrac{1}{3}$, $S_\infty = \dfrac{a}{1-r} = \dfrac{1}{\frac{4}{3}} = \dfrac{3}{4}$

Q1g $f(x) = \sqrt{x-8}$, $x - 8 \ge 0$, $x \ge 8$, dom $f(x)$ is $[8, \infty)$

Q2a By the quotient rule:

$$\frac{d}{dx}\left(\frac{\cos x}{x}\right) = \frac{x(-\sin x) - \cos x}{x^2} = \frac{-x \sin x - \cos x}{x^2}$$

Q2b $x^2 - x - 12 < 0$, $(x+3)(x-4) < 0$, $-3 < x < 4$



Q2c $y = \ln(3x)$, $m = \dfrac{dy}{dx} = \dfrac{1}{x}$. At $x = 2$, $m = \dfrac{1}{x} = \dfrac{1}{2}$

Q2di $\int \sqrt{5x+1}\,dx = \int (5x+1)^{\frac{1}{2}} dx = \dfrac{2(5x+1)^{\frac{3}{2}}}{15} + c$

Q2dii Let $u = 4 + x^2$, $\dfrac{1}{2}\dfrac{du}{dx} = x$

$$\int \frac{x}{4+x^2}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln\left(4+x^2\right) + c$$

Q2e $\int_0^6 (x+k)dx = 30$, $\left[\dfrac{(x+k)^2}{2}\right]_0^6 = 30$, $\dfrac{(6+k)^2}{2} - \dfrac{k^2}{2} = 30$,

$(6+k)^2 - k^2 = 60$, $6(6+2k) = 60$, $k = 2$

Q3ai Midpoint $M\left(\dfrac{-2+12}{2}, \dfrac{-4+6}{2}\right) = M(5,1)$

Q3aii Gradient of $BC = \dfrac{8-6}{6-12} = -\dfrac{1}{3}$

Q3aiii Gradient of $MN = \dfrac{2-1}{2-5} = -\dfrac{1}{3}$

.: *MN* and *BC* are parallel

.: corresponding angles $\angle AMN = \angle ABC$ and corresponding angles $\angle ANM = \angle ACB$

Hence $\Delta AMN, \Delta ABC$ are similar.

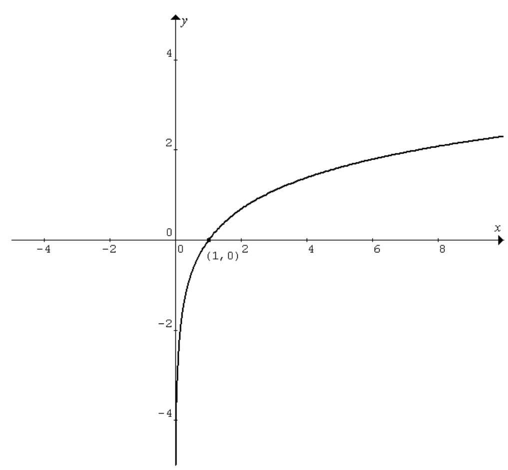
Q3aiv Equation of *MN*: $y - 2 = -\dfrac{1}{3}(x-2)$, $y = -\dfrac{1}{3}x + \dfrac{8}{3}$ or $x + 3y = 8$

Q3av Length of $BC = \sqrt{(12-6)^2 + (6-8)^2} = \sqrt{40} = 2\sqrt{10}$

Q3avi Let *h* be the perpendicular distance from *A* to *BC*.

$\dfrac{1}{2} \times 2\sqrt{10} \times h = 44$, $h = \dfrac{44}{\sqrt{10}} = \dfrac{22\sqrt{10}}{5}$

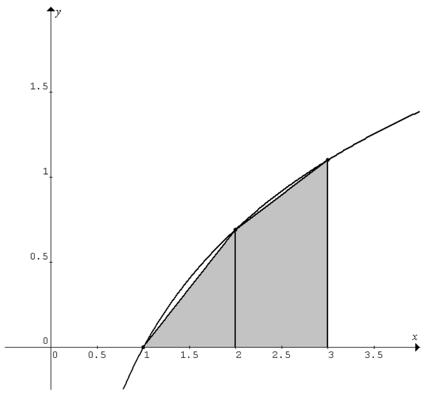
Q3bi $y = \ln x$



Q3bii $x = 1$, $y = 0$; $x = 2$, $y = \ln 2$; $x = 3$, $y = \ln 3$

$$\int_1^3 \ln x\,dx \approx \frac{1}{2}(0 + 2\ln 2 + \ln 3)(1) = \ln 2 + \frac{1}{2}\ln 3$$

Q3biii The approximation (total area of the two trapeziums) is less than the exact value of $\int_1^3 \ln x dx$ (the area under $y = \ln x$ between $x = 1$ and $x = 3$).



Q4ai Arithmetic sequence:
$a = 1$, $d = 0.75$, $t_9 = 1 + 8 \times 0.75 = 7$
.: 7 km in the 9th week

Q4aii $t_n = 1 + (n-1)0.75 = 10$, $n = 13$, .: the 13th week

Q4aiii $S_{13} = \frac{13}{2}(2(1) + (13-1)0.75) = 71.5$
Total distance $= 71.5 + 13 \times 10 = 201.5$ km

Q4b Area of the enclosed region $= \int_0^2 (e^{2x} - e^{-x}) dx$

$= \left[\frac{e^{2x}}{2} - \frac{e^{-x}}{-1}\right]_0^2 = \left(\frac{e^4}{2} + e^{-2}\right) - \left(\frac{1}{2} + 1\right) = \frac{e^4}{2} + e^{-2} - \frac{3}{2}$

Q4ci $\Pr(both.m.centres) = \frac{^4C_2}{^{12}C_2} = \frac{6}{66} = \frac{1}{11}$

Q4cii $\Pr(same.centre) = 3 \times \frac{1}{11} = \frac{3}{11}$

Q4ciii $\Pr(different.centres) = 1 - \frac{3}{11} = \frac{8}{11}$

Q4d $f(x) = 1 + e^x$
$f(x) \times f(-x) = (1 + e^x)(1 + e^{-x}) = 1 + e^x + e^{-x} + 1 = f(x) + f(-x)$

Q5ai $V = \pi r^2 h = 10$, .: $h = \frac{10}{\pi r^2}$
.: $A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{10}{\pi r^2}\right) = 2\pi r^2 + \frac{20}{r}$

Q5aii As $r \to 0^+$, $A \to \infty$; as $r \to \infty$, $A \to \infty$.
Also $A = 2\pi r^2 + \frac{20}{r}$ is a continuous function of r for $r > 0$.
.: A has a minimum value.
Let $\frac{dA}{dr} = 4\pi r - \frac{20}{r^2} = 0$, .: $r^3 = \frac{5}{\pi}$, $r = \sqrt[3]{\frac{5}{\pi}} \approx 1.17$ m
The minimum value occurs when $r = \sqrt[3]{\frac{5}{\pi}} \approx 1.17$ m.

Q5bi $RHS = \frac{1 + \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}$
$= \left(\frac{1}{\cos x}\right)^2 + \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \sec^2 x + \sec x \tan x = LHS$

Q5bii $\sec^2 x + \sec x \tan x$
$= \frac{1 + \sin x}{\cos^2 x} = \frac{1 + \sin x}{1 - \sin^2 x} = \frac{1 + \sin x}{(1 - \sin x)(1 + \sin x)} = \frac{1}{1 - \sin x}$

Q5biii $\int_0^{\frac{\pi}{4}} \frac{1}{1 - \sin x} dx = \int_0^{\frac{\pi}{4}} (\sec^2 x + \sec x \tan x) dx$
$[\tan x + \sec x]_0^{\frac{\pi}{4}} = \left(\tan\frac{\pi}{4} + \sec\frac{\pi}{4}\right) - (\tan 0 + \sec 0)$
$= 1 + \sqrt{2} - 1 = \sqrt{2}$

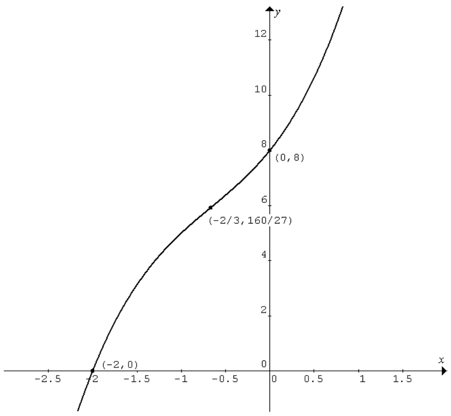
Q5c $\int_a^1 \frac{1}{x} dx = [\ln x]_a^1 = \ln 1 - \ln a = -\ln a = 1$, $a = e^{-1}$
$\int_1^b \frac{1}{x} dx = [\ln x]_1^b = \ln b - \ln 1 = \ln b = 1$, $b = e$

Q6ai $f(x) = (x + 2)(x^2 + 4)$,
$f'(x) = (x + 2)(2x) + (x^2 + 4) = 3x^2 + 4x + 4 \neq 0$ because
$\Delta = 4^2 - 4(3)(4) < 0$
.: $f(x)$ has no stationary points.

Q6aii $f''(x) = 6x + 4 = 0$, $x = -\frac{2}{3}$.
The graph of $y = f(x)$ is concave down for $x < -\frac{2}{3}$, and it is concave up for $x > -\frac{2}{3}$.

Q6aiii x-intercept: Let $y=0$, $(x+2)(x^2+4)=0$, .: $x=-2$

y-intercept: Let $x=0$, $y=(0+2)(0^2+4)=8$



Q6bi $9=5\angle POQ$, .: $\angle POQ=\frac{9}{5}=1.8$

Q6bii $OP=OQ=radius$

OT is a common side for ΔOPT and ΔOQT

$\angle OPT=\angle OQT=90°$, .: $PT=QT$ (Pythagoras)

.: ΔOPT and ΔOQT are congruent because they have equal corresponding sides.

Q6biii $\angle POT=\frac{1.8}{2}=0.9$, length of $PT=5\tan(0.9)\approx 6.30$ cm

Q6biv Area of the shaded region

$=5\times 6.3-\frac{1.8}{2\pi}\times\pi\times 5^2\approx 9.00$ cm^2

Q7ai $\ddot{x}=4\cos 2t$, $\dot{x}=1$ and $x=0$ when $t=0$

.: $\dot{x}=\int_0^t 4\cos 2t\,dt+1=[2\sin 2t]_0^t+1=2\sin 2t+1$

Q7aii Let $\dot{x}=2\sin 2t+1=0$,

$\sin 2t=-0.5$, $2t=\frac{7\pi}{6}$, .: $t=\frac{7\pi}{12}$

Q7aiii $\dot{x}=2\sin 2t+1$,

$x=\int_0^t(2\sin 2t+1)dt=[-\cos 2t+t]_0^t=1-\cos 2t+t$

Q7bi $y=x^2$: At $A(-1,1)$, $m=\frac{dy}{dx}=2x=-2$

Equation of the tangent: $y-1=-2(x+1)$, $2x+y=-1$

Q7bii Midpoint of $AB=M\left(\frac{-1+2}{2},\frac{1+4}{2}\right)=M\left(\frac{1}{2},\frac{5}{2}\right)$

Gradient of $AB=\frac{4-1}{2-(-1)}=1$

At C, $m=\frac{dy}{dx}=2x=1$, .: $x=\frac{1}{2}$

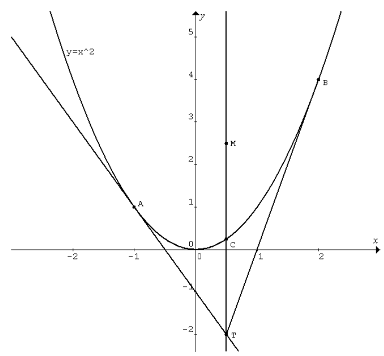
M and C have the same x-coordinate, .: MC is vertical.

Q7biii At $B(2,4)$, $m=\frac{dy}{dx}=2x=4$

Tangent at A: $2x+y=-1$; line MC: $x=\frac{1}{2}$

.: intersection $T\left(\frac{1}{2},-2\right)$

Gradient of $BT=\frac{4-(-2)}{2-\frac{1}{2}}=\frac{6}{\frac{3}{2}}=4$, same as the gradient of the tangent at B, .: line BT is a tangent to the parabola.



Q8a Let 1935 be $t=0$, and P is in millions.

$t=0$, $P=0.000102$; $t=75$, $P=200$; $\frac{dP}{dt}=kP$

$\frac{dt}{dP}=\frac{1}{k}\times\frac{1}{P}$, $t=\frac{1}{k}\int\frac{1}{P}dP$, $kt=\ln P+c$

$t=0$, $P=0.000102$, .: $0=\ln 0.000102+c$, $c=-\ln 0.000102$

.: $kt=\ln\frac{P}{0.000102}$

$t=75$, $P=200$, .: $75k=\ln\frac{200}{0.000102}$, $k=0.193185$

.: $0.193185t=\ln\frac{P}{0.000102}$

.: $P=0.000102e^{0.193185t}$

In 2035, $t=100$, $P\approx 25033$ millions

Q8b $\Pr(2heads)=[\Pr(head)]^2=0.36$

.: $\Pr(head)=0.6$, $\Pr(tail)=1-0.6=0.4$

.: $\Pr(2tails)=[\Pr(tail)]^2=0.16$

Q8ci $y=A\sin bx$, amplitude $A=4$

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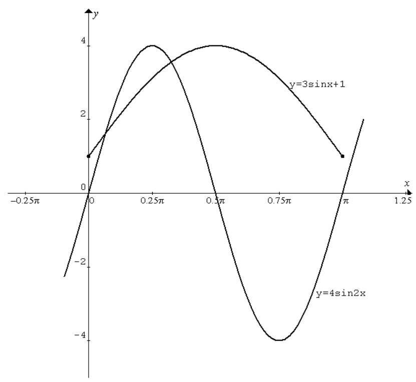
Q8cii Period $T = \frac{2\pi}{b} = \pi$, $b = 2$

Q8ciii



Q8d $f(x) = x^3 - 3x^2 + kx + 8$, $f'(x) = 3x^2 - 6x + k$

$f(x)$ is an increasing function, $f'(x) > 0$, when the discriminant of $3x^2 - 6x + k$ is less than 0, i.e. $\Delta = 36 - 12k < 0$, $k > 3$

Q9ai The balance after *n* deposits is given by

$B_n = \frac{Q(R^n - 1)}{R - 1}$ where $Q = \$500$ and $R = 1.005$

.: $B_{240} = \frac{500(1.005^{240} - 1)}{0.005} = 231020.4476$

.: at the end of the 240th month,

$\$P = 231020.4476 \times 1.005 = \$232175.55$

Q9aii(1) $A_n = PR^n - \frac{Q(R^n - 1)}{R - 1}$ where $Q = \$2000$ and $R = 1.005$

.: $A_n = P \times 1.005^n - \frac{2000(1.005^n - 1)}{0.005}$

$= P \times 1.005^n - 400000(1.005^n - 1)$

$= (P - 400000) \times 1.005^n + 400000$

Q9aii(2) Let $A_n = (P - 400000) \times 1.005^n + 400000 = 0$

.: $1.005^n = \frac{-400000}{P - 400000} = \frac{-400000}{232175.55 - 400000} = 2.383443$

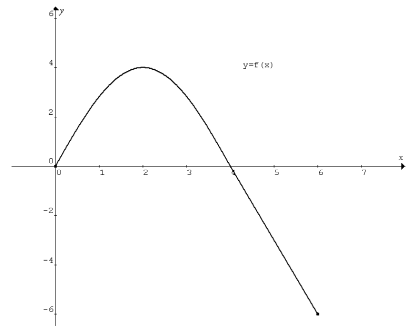
$n = \frac{\ln 2.383443}{\ln 1.005} \approx 174.14$, .: there will be money left in the account 174 months after retirement.

Q9bi $f'(x) > 0$, i.e. $f(x)$ is increasing for $0 \le x < 2$

Q9bii Given $f(0) = 0$, max $f(x) = f(2) = \int_0^2 f'(x)dx = A_1 = 4$

Q9biii $f(6) = A_1 - A_2 - 3 \times 2 = -6$

Q9biv



Q10ai $\angle DAC = \angle CAB$ (common angle)

.: $\angle DCA = \angle CBA$ (both triangles are isosceles)

.: $\angle ADC = \angle ACB = \theta$

.: ΔABC and ΔACD are similar.

Q10aii ΔABC and ΔACD are similar,

.: $\frac{x}{a} = \frac{a + y}{x}$, $x^2 = a(a + y)$, .: $x^2 = a^2 + ay$

Q10aiii Apply the cosine rule to ΔABC:

$(a + y)^2 = x^2 + x^2 - 2x^2 \cos\theta$

.: $(a + y)^2 = 2x^2(1 - \cos\theta)$, $(a + y)^2 = 2a(a + y)(1 - \cos\theta)$

$a + y = 2a(1 - \cos\theta)$, .: $y = a(1 - 2\cos\theta)$

Q10aiv Maximum of $1 - 2\cos\theta$ is 3 when $\theta = -\pi$, .: $y \le 3a$

Q10bi $x^2 + y^2 = r^2$, .: $y^2 = r^2 - x^2$

$V = \int_{r\sin\theta}^{r} \pi y^2 dx = \int_{r\sin\theta}^{r} \pi(r^2 - x^2)dx = \pi\left[r^2 x - \frac{x^3}{3}\right]_{r\sin\theta}^{r}$

$= \pi\left(\frac{2r^3}{3} - r^3\sin\theta + \frac{r^3\sin^3\theta}{3}\right) = \frac{\pi r^3}{3}(2 - 3\sin\theta + \sin^3\theta)$

Q10ii(1) Original depth $= r$ m, final depth $= \frac{r}{2}$ m at angle θ

.: $r\sin\theta = \frac{r}{2}$, $\theta = \frac{\pi}{6}$

Q10ii(2) Original volume $= \frac{1}{2} \times \frac{4\pi r^3}{3} = \frac{2\pi r^3}{3}$

Final volume $= \frac{\pi r^3}{3}\left(2 - 3\sin\frac{\pi}{6} + \sin^3\frac{\pi}{6}\right) = \frac{\pi r^3}{3}\left(\frac{5}{8}\right)$

Fraction $= \frac{\frac{5}{8} \times \frac{\pi r^3}{3}}{\frac{2\pi r^3}{3}} = \frac{5}{16}$

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