2006 VCAA Mathematical Methods Exam 2 Solutions

SECTION 1: Multiple-choice questions

1	2	3	4	5	6	7	8	9	10	11
C	B	A	B	D	D	B	E	B	A	D

12	13	14	15	16	17	18	19	20	21	22
C	E	A	D	A	A	D	A	E	D	C

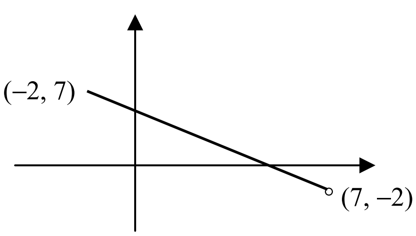
Q1 $y = x^2$; after translated 3 units down, $y = x^2 - 3$; after translated 2 units to the right, $y = (x-2)^2 - 3$.

Q2 $\tan(2x) = 1$, $2x = \tan^{-1}(1)$, $2x = ..., -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, ...$

$\therefore x = ..., -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}, ...$

The smallest positive value of *x* is $\frac{\pi}{8}$.

Q3 The graph of *f* is shown below.

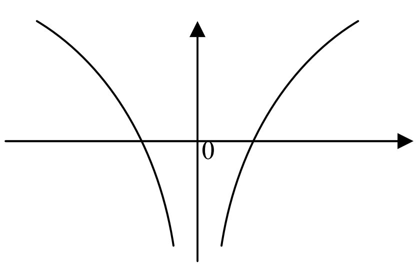


The range of *f* is $(-2, 7]$.

Q4 The range of *f* is $[-3+4,\ 3+4]$, i.e. $[1,\ 7]$.

Q5 Sampling without replacement:
$\Pr(yyy) = \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} = \frac{7}{24}$.

Q6 The graph of $y = \log_e(x^4)$ is shown below. It is a many-to-one function.



For $f : [a, \infty) \to R$, $f(x) = \log_e(x^4)$ to have an inverse function, it must be a one-to-one function, $\therefore a > 0$.

Q7 $x - b \neq 0$, $\therefore x \neq b$. Domain is $R \setminus \{b\}$.

Q8 $y = \log_3(x)$; after reflection in the *x*-axis, $y = -\log_3(x)$; after translation by 5 units up and translation 2 units right, $y = -\log_3(x-2) + 5$.

Q9 $y = 3a^{2x} + b$, $a^{2x} = \frac{y-b}{3}$, $2x = \log_a\left(\frac{y-b}{3}\right)$,

$x = \frac{1}{2}\log_a\left(\frac{y-b}{3}\right)$.

Q10 Given that $\frac{dr}{dt} = 3$ cm min^{-1} and $V = \frac{4}{3}\pi r^3$, when $r = 6$,

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(6)^2(3) = 432\pi$.

Q11 Either $2k+1 = k+1$ or $-(2k+1) = k+1$,

$\therefore$ either $k = 0$ or $k = -\frac{2}{3}$.

Q12 Binomial distribution: $n = 10$, X number of successes (heads), $p = 0.5$.
$\Pr(X \geq 8) = 1 - \Pr(X \leq 7) = 1 - binomcdf(10, 0.5, 7) = 0.0547$

Q13 Equation of inverse is $x = \frac{2}{3y+6} - 1$, $\therefore 3y + 6 = \frac{2}{x+1}$,

$3y = \frac{2}{x+1} - 6$, $\therefore y = \frac{1}{3}\left(\frac{2}{x+1} - 6\right) = \frac{2}{3x+3} - 2$.

$\therefore f^{-1}(x) = \frac{2}{3x+3} - 2$.

Q14 The graph has a negative gradient when $-3 < x < 3$.

Q15 Total area $= \left|\int_{-1}^{1} f(x)dx\right| + \left|\int_{1}^{4} f(x)dx\right| + \left|\int_{4}^{6} f(x)dx\right|$

$= -\int_{-1}^{1} f(x)dx + \int_{1}^{4} f(x)dx - \int_{4}^{6} f(x)dx$.

Q16 $f'(x) = g'(x) + 3$,
$f(x) = \int (g'(x) + 3)dx = g(x) + 3x + c$,
$\therefore f(0) = g(0) + c$, $\therefore 2 = 1 + c$, $c = 1$.
$\therefore f(x) = g(x) + 3x + 1$.

Q17 The quotient rule:
$\frac{d}{dx}\left(\frac{\sin(3x)}{2e^x - x}\right) = \frac{(2e^x - x)(3\cos(3x)) - (\sin(3x))(2e^x - 1)}{(2e^x - x)^2}$
$= \frac{3(2e^x - x)\cos(3x) - (2e^x - 1)\sin(3x)}{4e^{2x} - 4xe^x + x^2}$.

Q18 $a+b+0.4=1$ and $(-1)a+(0)b+(1)(0.4)=0.3$.
$\therefore a=0.1$ and $b=0.5$

Q19 $\int_0^k \left(1+e^{\frac{x}{k}}\right)dx=1$, $\left[x+ke^{\frac{x}{k}}\right]_0^k=1$, $\therefore (k+ke)-(k)=1$,
$ke=1$, $\therefore k=\frac{1}{e}=e^{-1}$.

Q20 Let $y=f(\sin(4x))$ and $u=\sin(4x)$, $\therefore y=f(u)$.
$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=f'(u)\times 4\cos(4x)=4\cos(4x)f'(\sin(4x))$.

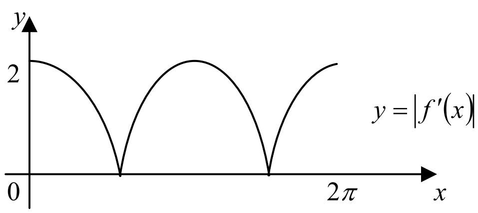
Q21 $\Pr(t<208)=normalcdf(-E99,208,200,10)\approx 0.788$

Q22 $y=a\log_e(x-b)$, $a<0$ and $b>0$, is the reflection of $y=\log_e x$ in the *x*-axis followed by a translation of *b* units to the right.

SECTION 2:

Q1ai $f(x)=2\sin(x)$, $f'(x)=2\cos(x)$.

Q1aii The graph of $y=|f'(x)|=|2\cos(x)|$ is shown below.



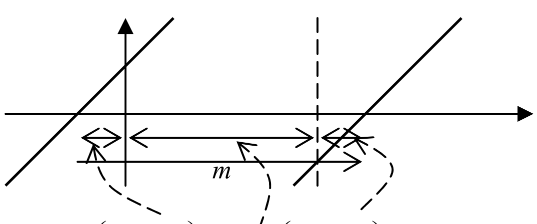
$|f'(x)|_{\min}=0$, $|f'(x)|_{\max}=2$.

Q1bi Refer to the two tangents shown in the given graph,
$x=2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$.

Q1bii When $x=\frac{\pi}{3}$, $y=2\sin\left(\frac{\pi}{3}\right)=\sqrt{3}$.
$\therefore y-\sqrt{3}=1\left(x-\frac{\pi}{3}\right)$, $y=x-\frac{\pi}{3}+\sqrt{3}$.

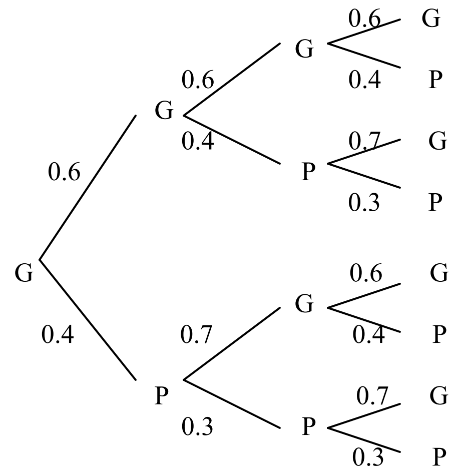
Q1biii *y*-intercept: $x=0$, $y=-\frac{\pi}{3}+\sqrt{3}$, $\left(0,-\frac{\pi}{3}+\sqrt{3}\right)$.
x-intercept: $y=0$, $x=\frac{\pi}{3}-\sqrt{3}$, $\left(\frac{\pi}{3}-\sqrt{3},0\right)$.

Q1c



By symmetry, $m=\left(\sqrt{3}-\frac{\pi}{3}\right)+2\pi+\left(\sqrt{3}-\frac{\pi}{3}\right)=2\sqrt{3}+\frac{4\pi}{3}$.

Q2a

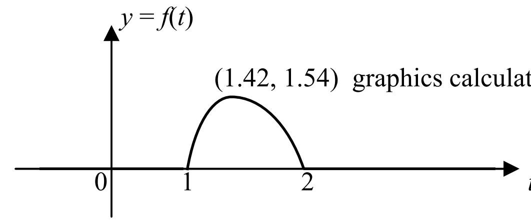


Q2ai $\Pr(PPP)=0.4\times0.3\times0.3=0.036$

Q2aii $\Pr(GPP\cup PGP\cup PPG)$
$=0.6\times0.4\times0.3+0.4\times0.7\times0.4+0.4\times0.3\times0.7=0.268$

Q2b $\Pr(X<50)=\Pr\left(Z<\frac{50-60}{\sigma}\right)=0.20$,
$\frac{-10}{\sigma}=invNorm(0.20)=-0.8416$,
$\therefore \sigma=11.9$ minutes

Q2c



Q2d 75 min = 1 h 15 min = 1.25 h
$\Pr(t<1.25)=\int_1^{1.25} f(t)dt=0.191$ graphics calculator.

Q2e $\Pr(t>1.25)=1-0.191=0.809$.
$n=5$, $p=0.809$, $\Pr(X=4)=binompdf(5,0.809,4)=0.41$.

Q2f Let *m* be the median time, $1\le m\le 2$.

$\int_1^m \left(4t^3-24t^2+44t-24\right)dt=0.5$,
$\left[t^4-8t^3+22t^2-24t\right]_1^m=0.5$,
$\left(m^4-8m^3+22m^2-24m\right)-(1-8+22-24)=0.5$
$m^4-8m^3+22m^2-24m+8.5=0$.

Solve by graphics calculator, $m=1.4588\text{ h}\approx 88$ minutes

Q3a $y = 3 - e^x - e^{-x}$.

y-intercept: Let $x = 0$, $y = b = 3 - e^0 - e^0 = 1$.

Q3b x-intercepts: Let $y = 0$, $0 = 3 - e^x - e^{-x}$,

$e^x - 3 + e^{-x} = 0$, $\therefore \left(e^x - 3 + e^{-x}\right)e^x = 0$,

$\left(e^x\right)^2 - 3\left(e^x\right) + 1 = 0$, $\therefore e^x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} = \dfrac{3 \pm \sqrt{5}}{2}$.

Hence $x = \log_e\left(\dfrac{3 \pm \sqrt{5}}{2}\right)$. $\therefore a = \log_e\left(\dfrac{3 + \sqrt{5}}{2}\right)$.

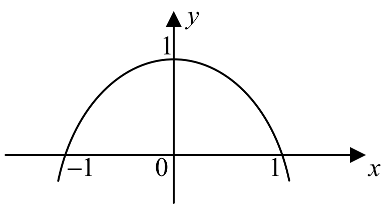
Q3ci

x	-0.5	0	0.5
y	0.74	1	0.74

Q3cii Estimated area $= 0.5 \times 0.74 + 0.5 \times 1 + 0.5 \times 0.74 \approx 1.2$ km^2.

Q3ciii $\$V = \$1.2mw$

Q3di



It is in the form $y = ax^2 + 1$. Use one of the x-intercepts, $(1,0)$ to find a: $0 = a(1)^2 + 1$, $a = -1$.

$\therefore y = 1 - x^2$.

Q3dii $\displaystyle\int_{-1}^{1}\left(1 - x^2\right)dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = 1.33$ km.

Q4ai Point B $(0,7)$ is part of $f(x) = px^3 + qx^2 + rx + s$,

$\therefore f(0) = s = 7$.

Q4aii $f'(x) = 3px^2 + 2qx + r$, $f'(0) = r = 4.25$.

Q4b $f(x) = px^3 + qx^2 + 4.25x + 7$ and

$f'(x) = 3px^2 + 2qx + 4.25$.

C $(1,9)$ is the furthest point (a turning point).

$\therefore f(1) = p + q + 4.25 + 7 = 9$ and $f'(1) = 3p + 2q + 4.25 = 0$,

i.e. $p + q = -2.25$ and $3p + 2q = -4.25$.

Q4c $f(x) = 0.25x^3 - 2.5x^2 + 4.25x + 7$

Use graphics calculator to find the zeros of $f(x)$, $x =$ 4 or 7.

$\therefore$D is $(4,0)$ and F is $(7,0)$.

Q4d Use graphics calculator to find $f'(x)$ at D, where $x = 4$.

$f'(4) = -3.75$.

Q4e Use graphics calculator to find the y-coordinate of point E, $y = -3.70$. $\therefore$ the greatest distance from the x-axis is 3.70 units.

Q4f $f'(x) = 0.75x^2 - 5x + 4.25$. Use graphics calculator to sketch $y = \left|f'(x)\right|$ and find its maximum value.

$\left|f'(x)\right|_{\max} = 4.1$.

Q4g $g(x) = \dfrac{a}{1 - bx}$, where $a, b > 0$, $g'(x) = \dfrac{ab}{(1 - bx)^2}$.

At point B $(0,7)$,

$g(0) = a = 7$ and $g'(0) = ab = 4.25$, $\therefore b = \dfrac{4.25}{a} = \dfrac{17}{28}$.

Q4h Area of shaded region A:

$$A = \int_{-2}^{0} \frac{7}{1 - \frac{17}{28}x}\,dx + \int_{0}^{4}\left(0.25x^3 - 2.5x^2 + 4.25x + 7\right)dx$$

$$= \left[\frac{7\log_e\left(1 - \frac{17}{28}x\right)}{-\frac{17}{28}}\right]_{-2}^{0} + \left[\frac{0.25x^4}{4} - \frac{2.5x^3}{3} + \frac{4.25x^2}{2} + 7x\right]_{0}^{4}$$

≈ 33.83 square units.

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