2006 VCAA Mathematical Methods (CAS) Sample exam 2 solutions

SECTION 1: Multiple-choice questions

1	2	3	4	5	6	7	8	9	10	11
A	C	A	A	D	A	B	A	D	A	A

12	13	14	15	16	17	18	19	20	21	22
D	E	A	C	C	A	B	B	A	C	E

Q1 $\dfrac{\int_0^\pi \sin x dx}{\pi} = \dfrac{[-\cos x]_0^\pi}{\pi} = \dfrac{2}{\pi}$ A

Q2 Determinant $\neq 0$, $(m-2)(m+2)-3\times 2 \neq 0$, $m \neq \sqrt{10}$ C

Q3 $|p+3|>3$, either $p+3>3$ or $-(p+3)>3$.
$\therefore p>0$ or $p+3<-3$, $\therefore p>0$ or $p<-6$. A

Q4 $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
Vertical dilation by a factor of 2, right by 3 units, up by 2 units
$\therefore y = x^2 \to y = 2(x-3)^2 + 2$ A

Q5 Use graphics calculator/CAS, $h = 0.5\left(1 - e^{-0.05t}\cos\left(\dfrac{3\pi t}{2}\right)\right)$,

evaluate $\dfrac{dh}{dt}$ at $t = 2.5$. $\dfrac{dh}{dt} \approx -1.45$.

By calculus, $\dfrac{dh}{dt} = 0.5\left(\dfrac{3\pi}{2}e^{-0.05t}\sin\left(\dfrac{3\pi t}{2}\right) + 0.05e^{-0.05t}\cos\left(\dfrac{3\pi t}{2}\right)\right)$.

At $t = 2.5$, $\dfrac{dh}{dt} \approx -1.45$. D

Q6 $N(t) = 1000e^{0.1t}$, $N(0) = 1000$, $N(10) = 1000e$.

Average rate $= \dfrac{1000e - 1000}{10} = 172$. A

Q7 $y = a(x+3)(x-1)(x-4)$, when $x = 0$, $y = 24$,
$\therefore 24 = a(3)(-1)(-4)$, $\therefore a = 2$. B

Q8 $[f(x)]^2 = f(y)$, $\left[e^{2x}\right]^2 = e^{2y}$, $4x = 2y$, $y = 2x$ A

Q9 The graph is the inverse of $y = x^3$, $\therefore y = \sqrt[3]{x} = x^{\frac{1}{3}}$ is the rule of the graph. D

Q10 $f(x) = \log_e(|x|) + 1$ is defined for $x \neq 0$. D is $R\setminus\{0\}$. A

Q11 $f(x) = 2x^3 - 3x^2 + 6$, to find the stationary points let $f'(x) = 6x^2 - 6x = 0$, $\therefore 6x(x-1) = 0$, $\therefore x = 0$ or $x = 1$.
For *f* to have an inverse function, *f* must be a one-to-one function, $\therefore a \geq 1$. A

Q12 From graph, $f(x) > 0$. Also $\int_0^0 f(x)dx = 0$.

$\therefore F(t) > 0$ for $t \in (0, b]$ D

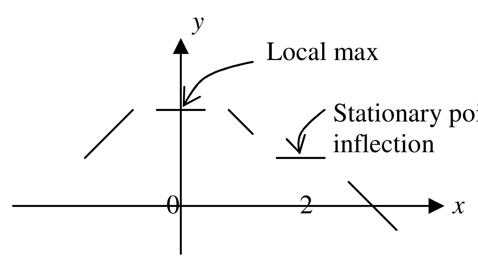
Q13 $\lim_{\delta x \to 0} \sum_{i=1}^{n} x_i \delta x = \int_0^4 x dx = \left[\dfrac{x^2}{2}\right]_0^4 = 8$ E

Q14 For $\pi < x < 2\pi$, $\sin(x)$ has a negative value,

$\therefore y = |\sin(x)| = -\sin(x)$ and $\dfrac{dy}{dx} = -\cos(x)$.

Hence at $x = k$, $\dfrac{dy}{dx} = -\cos(k)$. A

Q15



C

Q16 C

Q17 At $x = 4$, $y = 2x^{\frac{3}{2}} = 2\left(4^{\frac{3}{2}}\right) = 16$,

gradient of tangent $= \dfrac{dy}{dx} = 3x^{\frac{1}{2}} = 6$, gradient of normal $= -\dfrac{1}{6}$.

Equation of normal: $y - 16 = -\dfrac{1}{6}(x-4)$,

$\therefore y = -\dfrac{1}{6}x + \dfrac{50}{3}$. A

Q18 $f(x)$ is an increasing function, $\therefore f'(x) > 0$ for all *x*. Also as *x* increases, $f'(x)$ increases. B

Q19 $\Pr(X > 15) = \Pr\left(Z > \dfrac{15 - 12.2}{1.4}\right) = \Pr(Z > 2)$. B

Q20 Given $p = 0.15$, and $\Pr(X \ge 1) > 0.95$.

$\therefore 1 - \Pr(X = 0) > 0.95$, $\therefore \Pr(X = 0) < 0.05$,

$\therefore (1 - 0.15)^n < 0.05$, $0.85^n < 0.05$,

$n \log_{10} 0.85 < \log_{10} 0.05$, $-0.0706n < -1.3010$,

$\therefore n > 18.4$. A

Q21 $\Pr(00 \cup 11 \cup 22 \cup 33) = \Pr(00) + \Pr(11) + \Pr(22) + \Pr(33)$

$= 0.4^2 + 0.3^2 + 0.2^2 + 0.1^2 = 0.30$ C

Q22

$\Pr(X > a) = 0.25$, $\int_a^{\pi} \left(\frac{1}{2}\sin(x)\right)dx = 0.25$, $\left[-\frac{1}{2}\cos(x)\right]_a^{\pi} = 0.25$,

$-\frac{1}{2}\cos(\pi) + \frac{1}{2}\cos(a) = 0.25$, $1 + \cos(a) = 0.5$, $\cos(a) = -0.5$,

$\therefore a \approx 2.09$ E

SECTION 2

Q1ai $f(x) = ax^3 + bx^2 + cx + 2$, $x \in [0,2]$

$f'(x) = 3ax^2 + 2bx + c$

Turning point $(1,1)$, $f'(1) = 3a + 2b + c = 0$ ……(1)

and $f(1) = a + b + c + 2 = 1$ ……(2)

Solve (1) and (2) simultaneously for a and b in terms of c, $a = c + 2$ and $b = -2c - 3$.

Q1aii $f(x) = ax^3 + bx^2 + cx + 2$, $f(2) = 0$,

$\therefore 8(c + 2) + 4(-2c - 3) + 2c + 2 = 0$, $\therefore c = -3$.

Q1bi $f(x) = (x-1)^2(x-2) + 1$,

$f'(x) = (x-1)^2(1) + 2(x-1)(x-2) = (x-1)((x-1) + 2(x-2))$

$= (x-1)(3x-5)$.

Q1bii At the turning points, $f'(x) = 0$, $(x-1)(3x-5) = 0$,

$x = 1$ and $y = 1$ or $x = \frac{5}{3}$ and $y = \frac{23}{27}$.

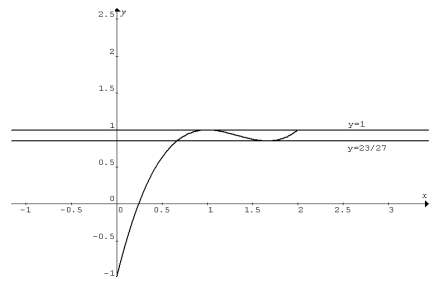
$\therefore m = 1$ and $n = \frac{5}{3}$.

Q1biii At end points: $f(0) = -1$, $f(2) = 1$

Local maximum: $f(1) = 1$; local minimum: $\left(\frac{5}{3}, \frac{23}{27}\right)$

$\therefore$ absolute maximum value of $f(x)$ is 1, and the absolute minimum value of $f(x)$ is -1.

Q1biv



For $(x-1)^2(x-2) + 1 = p$ to have exactly one solution, $-1 \le p < \frac{23}{27}$.

Q1ci Dilation from the y-axis by a scale factor of k, then downward translation by 1 unit.

Q1cii $y = f\left(\frac{x}{k}\right) - 1 = \left(\frac{x}{k} - 1\right)^2\left(\frac{x}{k} - 2\right) + 1 - 1 = \left(\frac{x}{k} - 1\right)^2\left(\frac{x}{k} - 2\right)$,

x-intercepts: $\left(\frac{x}{k} - 1\right)^2\left(\frac{x}{k} - 2\right) = 0$, $\therefore \frac{x}{k} - 1 = 0$ or $\frac{x}{k} - 2 = 0$.

Hence $x = k$ or $x = 2k$.

The x-intercepts are $(k,0)$ and $(2k,0)$.

Q1ciii $f(x+h) = 1$, $(x+h-1)^2(x+h-2) + 1 = 1$,

$\therefore (x+h-1)^2(x+h-2) = 0$.

To have exactly one positive value solution, $h - 1 \ge 0$ and $h - 2 < 0$. $\therefore h \ge 1$ and $h < 2$, i.e. $1 \le h < 2$.

Note: zero is neither positive nor negative.

Q2a $y = (2x^2 - 3x)e^{ax}$ passes through $(2,3)$.

$3 = (2(2^2) - 3(2))e^{2a}$, $3 = 2e^{2a}$, $\therefore 2a = \log_e 1.5$, $a = 0.5\log_e 1.5$.

Q2bi $a = 1$, $\therefore y = (2x^2 - 3x)e^x = x(2x-3)e^x$.

x-intercepts: $x = 0$ or $\frac{3}{2}$. The x-coordinate of A is $\frac{3}{2}$.

Q2bii Area $= -\int_0^{\frac{3}{2}} (2x^2 - 3x)e^{ax}dx = 10$

$a = 2.474$ by CAS.

Q2c At turning points, $\frac{dy}{dx} = 0$,

$\therefore \frac{dy}{dx} = e^x(2x^2 + x - 3) = e^x(2x+3)(x-1) = 0$.

Since $e^x \ne 0$, $\therefore x = -\frac{3}{2}$ or 1. Hence x-coordinate of B is 1 and y-coordinate is $-e$. $\therefore B$ is $(1,-e)$.

Q2di $\frac{dy}{dx} = e^x\left(2x^2 + x - 3\right)$

At the origin, $\frac{dy}{dx} = -3$.

Equation of the tangent at the origin is $y = -3x$.

Q2dii $y = \left(2x^2 - 3x\right)e^x$ and $y = -3x$ intersects at D.

By graphics calculator/CAS, D is $(0.87, -2.62)$.

Q3a From the given transition matrix, $\Pr(N \mid S) = \frac{4}{5}$.

Q3b Let the state matrix be $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ on Monday night.

On Thursday night the state matrix is $\begin{bmatrix} \frac{2}{5} & \frac{4}{5} \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{68}{125} \\ \frac{57}{125} \end{bmatrix}$.

.: Pr(N on Thursday night) is $\frac{68}{125}$.

Q3c $\begin{bmatrix} \frac{2}{5} & \frac{4}{5} \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix}^n \to \begin{bmatrix} 0.571428... & 0.571428... \\ 0.428571... & 0.428571... \end{bmatrix}$ as $n \to \infty$, by CAS/graphics calculator.

.: in the long term 57% approx. of nights the fox will hunt on the north side of the creek.

Q3d $\Pr(t > 3) = \int_3^4 \left(\frac{3}{32}t(4-t)\right)dt = \left[\frac{3}{32}\left(2t^2 - \frac{t^3}{3}\right)\right]_3^4$

$= \frac{3}{32}\left(2(4^2) - \frac{4^3}{3}\right) - \frac{3}{32}\left(2(3^2) - \frac{3^3}{3}\right) = \frac{5}{32}$.

Q3e Binomial, $p = \frac{5}{32}$, $n = 3$.

$\Pr(X \geq 2) = 1 - \Pr(X \leq 1) = 1 - 0.934 = 0.066$.

Q3f $\Pr\left(t < \frac{n}{60}\right) = \int_0^{\frac{n}{60}} \left(\frac{3}{32}t(4-t)\right)dt = \left[\frac{3}{32}\left(2t^2 - \frac{t^3}{3}\right)\right]_0^{\frac{n}{60}}$

$= \frac{3}{32}\left(2\left(\frac{n}{60}\right)^2 - \frac{1}{3}\left(\frac{n}{60}\right)^3\right) = \frac{1}{32}\left(\frac{n^2}{600} - \frac{n^3}{216000}\right)$.

$\therefore \frac{1}{32}\left(\frac{n^2}{600} - \frac{n^3}{216000}\right) = 0.104$. Use CAS/graphics calculator to solve this equation. $n = 48$.

Q4a Since $-1 \leq \sin\left(\frac{(5t-1)\pi}{2}\right) \leq 1$

$\therefore$ max height = 62 + 60 = 122 m

Q4b min height = 62 – 60 = 2 m

Q4c Period $T = \frac{2\pi}{n} = \frac{2\pi}{\frac{5\pi}{2}} = 0.8$ hour, i.e. 48 minutes.

$\therefore$ At 1.48 pm.

Q4di $92 = 62 + 60\sin\left(\frac{(5t-1)\pi}{2}\right)$, $\therefore \sin\left(\frac{(5t-1)\pi}{2}\right) = \frac{1}{2}$,

$\frac{(5t-1)\pi}{2} = \frac{\pi}{6}, \frac{5\pi}{6}$.

Hence $t = \frac{4}{15}$ (first time), $\frac{8}{15}$ (second time).

$t = \frac{4}{15}$ hour $= 16$ minutes, $\therefore$ at 1.16 pm.

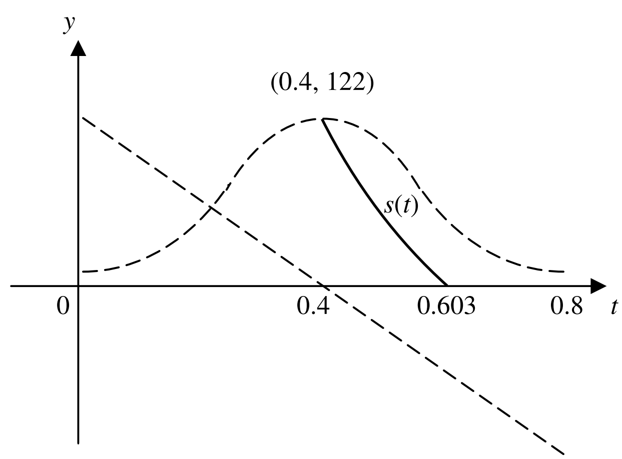
Q4dii At least 92 metres above ground level when $\frac{4}{15} \leq t \leq \frac{8}{15}$,

$\therefore \Delta t = \frac{8}{15} - \frac{4}{15} = \frac{4}{15}$ hour, i.e. 16 minutes.

Q4ei $\frac{dh}{dt} = 60 \times \frac{5\pi}{2}\cos\left(\frac{(5t-1)\pi}{2}\right) = 150\pi\cos\left(\frac{(5t-1)\pi}{2}\right)$ mh^{-1}.

Q4eii When $t = 1$, $\frac{dh}{dt} = 150\pi\cos(2\pi) = 150\pi = 471.2$ mh^{-1}.

Q4f i and ii



Domain for $s(t)$ is $[0.4, 0.603]$.

↑

CAS/graphics calculator

Q4fiii Spider reaches ground when $s(t) = 0$,

i.e. at $t = 0.603$ hour.

At the highest point, $t = 0.4$ hour.

$\therefore \Delta t = 0.603 - 0.4 = 0.203$ hour, i.e. 12 minutes.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors